On Performance of Sphere Decoding and Markov Chain Monte Carlo Detection Methods

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***Abstract*—In a recent work, it has been found that the suboptimum detectors that are based on Markov chain Monte Carlo (MCMC) simulation techniques perform significantly better than their sphere decoding (SD) counterparts. In this letter, we explore the sources of this difference and show that a modification to existing sphere decoders can result in some improvement in their performance, even though they still fall short when compared with the MCMC detector. We also present a novel SD detector that is an exact realization of max-log-MAP detector. We call this exact max-log SD detector. Comparison of the results of this detector with those of the max-log version of the MCMC detector reveals that the latter is near optimal.**

***Index Terms*—Detection, Markov chain Monte Carlo (MCMC), multiple-input multiple-output (MIMO), sphere decoding (SD).**

I. Introduction

TWO major research activities have dominated the design of power and bandwidth-efficient wireless communication systems in recent years: 1) communication through multiple antennas, known as multiple-input multiple-output (MIMO) communication, and 2) iterative decoding/detection techniques. MIMO communication allows simultaneous transmission of multiple symbols from multiple transmit antennas. This results in a linear increase in the channel capacity proportional to the number of transmit antennas, when there are sufficient numbers of receive antennas. Iterative decoding/detection, on the other hand, is a feasible method that greatly improves the bit-error-rate (BER) performance of communication systems, bringing them very close to the Shannon channel capacity.

The combination of MIMO and iterative decoding/detection techniques has naturally been studied as a means of approaching the capacity of MIMO channels. Fig. 1 presents a block diagram of an MIMO receiver that combines works based on this principle [1], [2]. Here, the MIMO channel plays the role of an inner code. The outer code is a channel code that may be a convolutional code or a more advanced turbo or low-density parity-check (LDPC) code. We assume an MIMO channel with M transmit antennas and N receive antennas and a flat fading model

$$\mathbf{y} = \mathbf{H}\mathbf{d} + \mathbf{n} \tag{1}$$

where $\mathbf{d} = [d_1\ d_2 \cdots d_M]^{\mathrm{T}}$ is a vector of transmitted symbols, $\mathbf{H}$ is the channel gain matrix (of size $N \times M$), $\mathbf{n}$ is a vector of channel additive noise, $\mathbf{y}$ is the received signal vector, and the superscript T denotes transpose.

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Hochwald and ten Brink [3] have followed the receiver structure of Fig. 1 and used sphere decoding (SD) as a soft-in soft-out MIMO detector and reported some results that approach the capacity of MIMO channels. Vikalo *et al.* [4] have also studied the use of SD and demonstrated similar results. The difference between the two approaches is that [3] uses a candidate list (a list of likely choices of $\mathbf{d}$ that matches the channel output $\mathbf{y}$) that is chosen independently of the *a priori* information from the channel decoder, while [4] includes this information in finding the list.

SD may be thought of as a method of searching the M-dimensional space spanned by $\mathbf{d}$ and choosing a list of candidates that match $\mathbf{y}$. In SD, this is done in a deterministic manner by finding samples of $\mathbf{d}$ that result in a small distance between $\mathbf{y}$ and $\mathbf{Hd}$. Gibbs sampling (GS), a statistical method based on Markov chain Monte Carlo (MCMC) simulation techniques [5], [6], is an alternative method that may be used for choosing a list of $\mathbf{d}$ with the same goal. In a recent work [8] (also see [7] for our initial results), we have proposed an MIMO receiver that follows the structure of Fig. 1 and applies GS to obtain samples of $\mathbf{d}$. The results presented in [8] are superior to those in [3] and [4]. In this letter, we explore the source of this performance gain and show that there are two contributers: 1) the search method for obtaining the samples of $\mathbf{d}$ and 2) the way the samples are handled to produce the LLR values.

II. Review of the Results From SD

Similar to the previous works [3] and [4], we assume that each vector of the transmitted symbols $\mathbf{d}$ is obtained, through a mapping, from a vector $\mathbf{b}$ of information bits $b_1, b_2, \cdots, b_{M\cdot M_c}$. We also recall from [4] that a candidate list $\mathcal{L}$ of $\mathbf{d}$ may be generated by picking samples that satisfy the inequality

$$(\mathbf{d} - \hat{\mathbf{d}})^{\dagger}\mathbf{H}^{\dagger}\mathbf{H}(\mathbf{d} - \hat{\mathbf{d}}) - 2\sigma^2 \ln P^{\mathrm{e}}(\mathbf{d}) < r^2 \tag{2}$$

where $\dagger$ denotes Hermitian, $P^{\mathrm{e}}(\mathbf{d})$ is the (extrinsic) probability of $\mathbf{d}$ according to the feedback from the channel decoder, $2\sigma^2$ is the variance of each element of $\mathbf{n}$, $\hat{\mathbf{d}} = (\mathbf{H}^{\dagger}\mathbf{H})^{-1}\mathbf{H}^{\dagger}\mathbf{y}$ is the center of the sphere, and r is its radius.

Given a candidate list $\mathcal{L}$, the following equation is used to obtain an estimate of the extrinsic LLR value of b_k [3], [4]:

$$\begin{aligned}\lambda_1^{\mathrm{e}}(b_k) = &\max_{\mathbf{d}\in\mathcal{L}_k^+}\left\{-\frac{1}{2\sigma^2}\|\mathbf{y} - \mathbf{Hd}\|^2 + \frac{1}{2}\mathbf{b}_{-k}^{\mathrm{T}}\boldsymbol{\lambda}_{2,-k}^{\mathrm{e}}\right\}\\ &- \max_{\mathbf{d}\in\mathcal{L}_k^-}\left\{-\frac{1}{2\sigma^2}\|\mathbf{y} - \mathbf{Hd}\|^2 + \frac{1}{2}\mathbf{b}_{-k}^{\mathrm{T}}\boldsymbol{\lambda}_{2,-k}^{\mathrm{e}}\right\}\end{aligned} \tag{3}$$

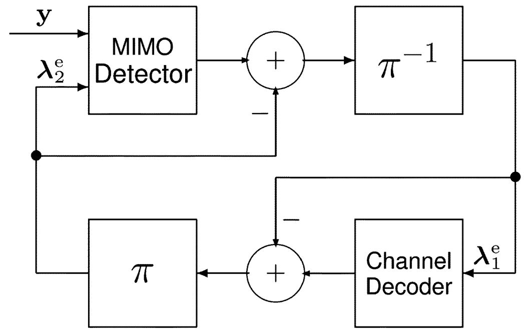

Fig. 1. Receiver structure. MIMO detector and channel decoder are soft-in soft-out blocks that exchange information $\boldsymbol{\lambda}_1^e$ and $\boldsymbol{\lambda}_2^e$ in a turbo loop.

where $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$ denote the subsets of $\mathcal{L}$ for which b_k is $+1$ and -1, respectively, $\mathbf{b}_{-k}$ is obtained from $\mathbf{b}$ by removing b_k, and $\boldsymbol{\lambda}_{2,-k}^e$ is the vector of the extrinsic LLR values of $\mathbf{b}_{-k}$ from the channel decoder.

SD adopts a tree search approach to obtain samples of $\mathbf{d}$. Fig. 2 presents an example of the tree search adopted in [3] and [4]. The search starts from a root node and begins with examining possible choices of d_1 that may satisfy (2). For each choice of d_1, the possible choices of d_2 are examined. The procedure continues for the rest of the elements of $\mathbf{d}$ similarly. Each choice of an element of $\mathbf{d}$ is indicated by a branch in the tree. Also, for convenience of demonstration, it is assumed that the elements of $\mathbf{d}$ are binary. The search collects samples of $\mathbf{d}$ by running through the branches of the tree from top to bottom and left to right. It is worth noting that many searches in the tree encounter nodes where no branches beyond them can lead to a point within the sphere, i.e., satisfy (2). In Fig. 2, these are indicated by bold nodes. We refer to these as *terminal nodes*.

III. Review of the Results From MCMC Detection

In the MCMC detection method, a list $\mathcal{L}$ of samples of $\mathbf{d}$ is generated by running one or more Markov chains that converge toward the conditional distribution

$$P(\mathbf{d}|\mathbf{y},\boldsymbol{\lambda}_2^e) = K \exp\left(-\frac{\|\mathbf{y}-\mathbf{Hd}\|^2}{2\sigma^2}\right) P^e(\mathbf{d}). \qquad (4)$$

This procedure clearly leads to generation of samples of $\mathbf{d}$ that result in small values for $\|\mathbf{y}-\mathbf{Hd}\|^2 - 2\sigma^2 \ln P^e(\mathbf{d})$. Hence, the candidate list generated by this procedure is similar to those generated through SD [see (2)].

GS is the common procedure of generating samples of $\mathbf{d}$ in MCMC detectors. It visits successive elements of $\mathbf{d}$, sequentially, by the following procedure:

- Initialize $\mathbf{d}^{(-N_b)}$ (randomly).
- For $n = -N_b + 1$ to N_s
  Draw $d_1^{(n)}$ from $P(d_1|d_2^{(n-1)},\ldots,d_M^{(n-1)},\mathbf{y},\boldsymbol{\lambda}_2^e)$
  Draw $d_2^{(n)}$ from $P(d_2|d_1^{(n)},d_3^{(n-1)},\ldots,d_M^{(n-1)},\mathbf{y},\boldsymbol{\lambda}_2^e)$
  $\vdots$
  Draw $d_M^{(n)}$ from $P(d_M|d_1^{(n)},\ldots,d_{M-1}^{(n)},\mathbf{y},\boldsymbol{\lambda}_2^e)$

In this procedure, $\mathbf{d}^{(-N_b)}$ is initialized randomly, taking into account the *a priori* information $\boldsymbol{\lambda}_2^e$. The first N_b iterations of the "for" loop, called burn-in period, is to let the Markov chain converge to near its stationary distribution. Hence, the samples of $\mathbf{d}$ used for LLR computations are $\mathbf{d}^{(n)} = [d_1^{(n)}\ d_2^{(n)} \cdots d_M^{(n)}]^T$, for $n = 1, 2, \cdots, N_s$. The samples generated by a single Markov chain are usually highly correlated [5]. In [8], it is noted that the performance of the MCMC detection method significantly improves if a parallel set of Markov chains is used. We adopt this strategy while generating the simulation results in Section VI.

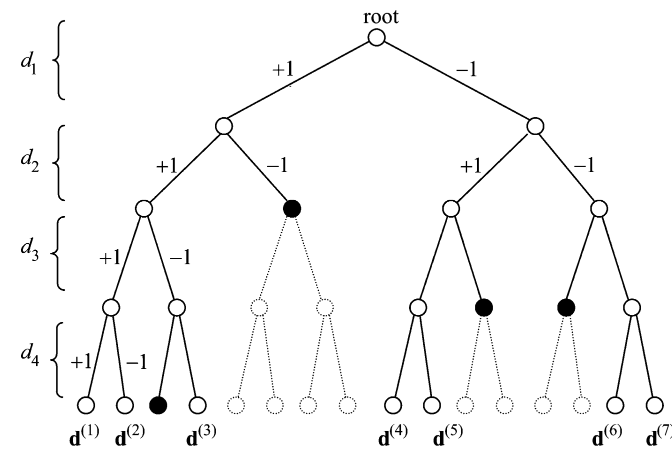

Fig. 2. Example of the tree search in SD.

From the above, we note that SD and GS are fundamentally similar in the sense that they both attempt to find sets of samples $\mathbf{d}$ that result in small values of $\|\mathbf{y}-\mathbf{Hd}\|^2 - 2\sigma^2 \ln P^e(\mathbf{d})$. However, [8] has adopted a different approach of handling the list $\mathcal{L}$. More particularly, the lists $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$, in (3), are, respectively, replaced by the expanded lists $\mathcal{U}_k^+$ and $\mathcal{U}_k^-$, which are obtained from $\mathcal{L}$ as follows. We note that in mapping $\mathbf{b}$ to $\mathbf{d}$, b_k is mapped to one of the elements of $\mathbf{d}$. The expansion begins with identifying the element of $\mathbf{d}$ to which the desired bit b_k is mapped. The set $\mathcal{L}$ is then expanded by giving this element of sample vectors $\mathbf{d}^{(n)}$ all possible values that it can take, from the symbol alphabet. This results in a set that we call $\mathcal{U}_k$. $\mathcal{U}_k^+$ and $\mathcal{U}_k^-$ are then generated as the subsets of $\mathcal{U}_k$ in which b_k is $+1$ and -1, respectively.

IV. Modified SD

Clearly, one can also propose an implementation of SD using the expanded lists $\mathcal{U}_k^+$ and $\mathcal{U}_k^-$ instead of $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$. The simulation results presented in Section VI show that this modification to SD improves its performance. We call this implementation of SD *modified SD*.

The improved behavior of the modified SD may be explained as follows. We recall from [3] that when a particular bit, say, b_k, is more likely to be $+1$ (or -1), the subset $\mathcal{L}_k^-$ (or $\mathcal{L}_k^+$) may be empty. In such cases, [3] suggests that $\lambda_1^e(b_k)$ be replaced by some extreme values, which, of course, may be inaccurate. Expansion of the lists, i.e., using $\mathcal{U}$ in place of $\mathcal{L}$, avoids such undesirable cases.

V. Exact Max-Log SD Detector

The exact max-log-MAP estimate of b_k is given by

$$\lambda_1^e(b_k) = \max_{\mathbf{d}\in\mathcal{D}_k^+}\left\{-\frac{1}{2\sigma^2}\|\mathbf{y}-\mathbf{Hd}\|^2 + \frac{1}{2}\mathbf{b}_{-k}^T\boldsymbol{\lambda}_{2,-k}^e\right\} - \max_{\mathbf{d}\in\mathcal{D}_k^-}\left\{-\frac{1}{2\sigma^2}\|\mathbf{y}-\mathbf{Hd}\|^2 + \frac{1}{2}\mathbf{b}_{-k}^T\boldsymbol{\lambda}_{2,-k}^e\right\} \qquad (5)$$

where $\mathcal{D}$ is a complete list of all choices of $\mathbf{d}$, and $\mathcal{D}_k^+$ and $\mathcal{D}_k^-$ are the subsets of $\mathcal{D}$ in which b_k is $+1$ and -1, respectively.

Clearly, the exponential growth of the size of $\mathcal{D}$ may prohibit direct realization of (5). This problem may be resolved by applying SD to each of the terms on the right-hand side of (5) separately. The true maximizer of the two terms on the right-hand side of (5) are within these subsets. This requires $2M \cdot M_c$ independent sphere decoders that may still be too complex to realize. Nonetheless, a study of such a detector is useful and can be considered as a benchmark for evaluation of other suboptimum detectors.

VI. Computer Simulation

We present the BER results for two MIMO systems: one with four transmit and four receive antennas and the other with eight transmit and eight receive antennas. Data symbols are chosen from a 16-point QAM constellation. The channel code is a rate 1/2 parallel concatenated (turbo) code with feedforward polynomial $1+D^2$ and feedback polynomial $1+D+D^2$. Each block of information bits has the length of 9216. The channel matrix $\mathbf{H}$ is independent over time and has complex i.i.d. entries. These parameters are similar to those of [3]. We also recall that the SD of [3] uses lists of length 512 and 1024 in the cases of four and eight antennas, respectively. For MCMC, L parallel Gibbs samplers each of length N_s and with no burning periods (i.e., $N_\mathrm{b} = 0$) are used. Because of the stochastic nature of MCMC, repeated samples of $\mathbf{d}^{(n)}$ (that should be discarded) occur, and this reduces the list length below the maximum length LN_s. For modified SD, we start with a small r and increase it gradually until we find a sufficient number of samples, say, $N_{\min}$ samples. Since there is no clear relationship between r and the number of choices of $\mathbf{d}$ that satisfy (2), one occasionally encounters a situation where the number of choices of $\mathbf{d}$ that satisfy (2) is excessively large. In such cases, we stop choosing samples of $\mathbf{d}$ when a certain number of samples, say, $N_{\max}$, is collected. We choose the parameters L and N_s of the MCMC detector and $N_{\min}$ and $N_{\max}$ of the modified SD detector such that the average list lengths in these cases are comparable with the list lengths used in the SD of [3].

The BER results of the case 4×4 are presented in Fig. 3. For the MCMC, $L = 10$ and $N_s = 10$. For the modified SD, $N_{\max} = 100$ and $N_{\min}$ is set equal to 25, 20, 15,10, and 5 for the first, second, third, forth, and fifth iterations of the turbo loop, respectively. For the sixth iteration onwards, $N_{\min}$ is set equal to one. These choices of $N_{\min}$ are made on the following basis. In the first iterations, when no prior information is available, any reasonable choice of r may lead to many choices of $\mathbf{d}$ that satisfy (2), and we need many samples of $\mathbf{d}$ for obtaining reasonable estimates of LLRs. In later iterations, the tree search is more controlled by the prior information. As a result, for given r, there exist only a small set of choices of $\mathbf{d}$ that satisfy (2) and a lower number of samples is sufficient for estimation of LLR values.

From the results of Fig. 3, we observe that the performance of the MCMC detector is very close to that of the exact max-log SD detector. The modified SD performs better than the original SD of [3]; however, it is not as good as the MCMC and the exact max-log SD detectors. This loss can be explained as follows.

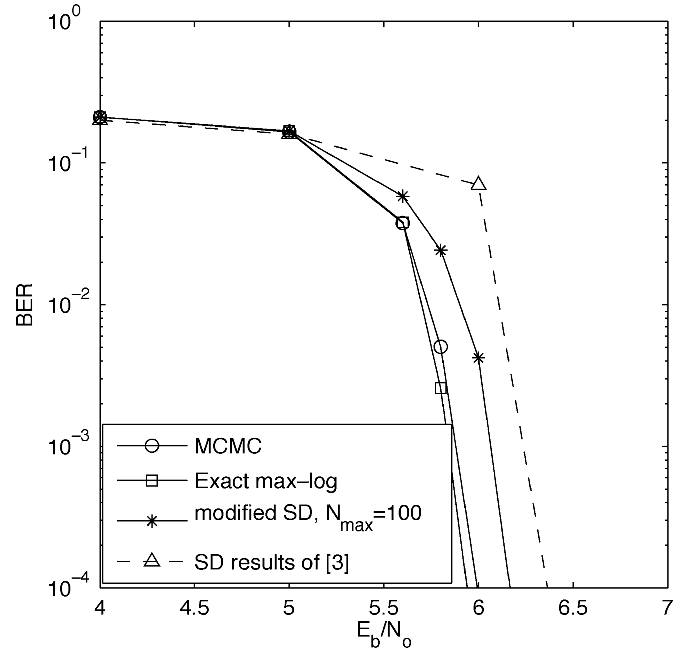

Fig. 3. BER results of the SD, modified SD, MCMC, and exact max-log SD detectors for an MIMO system with four transmit and four receive antennas.

SD adopts a tree search approach to obtain samples of $\mathbf{d}$. From the example given in Fig. 2, the following observation is made. The samples of $\mathbf{d}$ are chosen sequentially starting from the left side of the tree and moving to the right. When for a given r, the number of samples that satisfy (2) is larger than $N_{\max}$, some of these samples will be excluded from the list $\mathcal{L}$. This truncation of the samples results in a list $\mathcal{L}$ that has a noneven distribution around the center of the sphere $\hat{\mathbf{d}}$. In GS, on the other hand, such a noneven distribution is naturally avoided since elements of $\mathbf{d}$ are visited on a regular basis. The accuracy of this argument can be confirmed by removing the upper limit $N_{\max}$ in our simulations. The results (not shown here) are very similar to those of the MCMC detector.

The above experiment was repeated for an 8×8 MIMO system. The results are presented in Fig. 4. For the MCMC, $L = 20$ and $N_s = 20$. For the modified SD, two choices of $N_{\max} = 400$ and 800 are considered, and N_{min} is set equal to 50, 40, 30,20, and 10 for the first, second, third, forth, and fifth iterations, respectively. For the sixth iteration onwards, $N_{\min}$ is set equal to one. We are unable to present the results of the exact max-log SD detector because of its prohibitive complexity. Here, the MCMC detector outperforms the SD of [3] by about 1 dB. The modified SD also outperforms the SD of [3]. However, similar to the case of Fig. 3, it does not perform as well as the MCMC detector.

VII. Computational Complexity

Hassibi and Vikalo [11] have studied the computational complexity of SD and have shown that the widely cited polynomial complexity of SD is only valid when signal-to-noise ratio (SNR) is high. When SNR is low, the complexity of SD is predicted as exponential. Similar results have been reported in [12], where the authors have discussed a number of branch and bound (BBD) algorithms, including SD. The exponential complexity

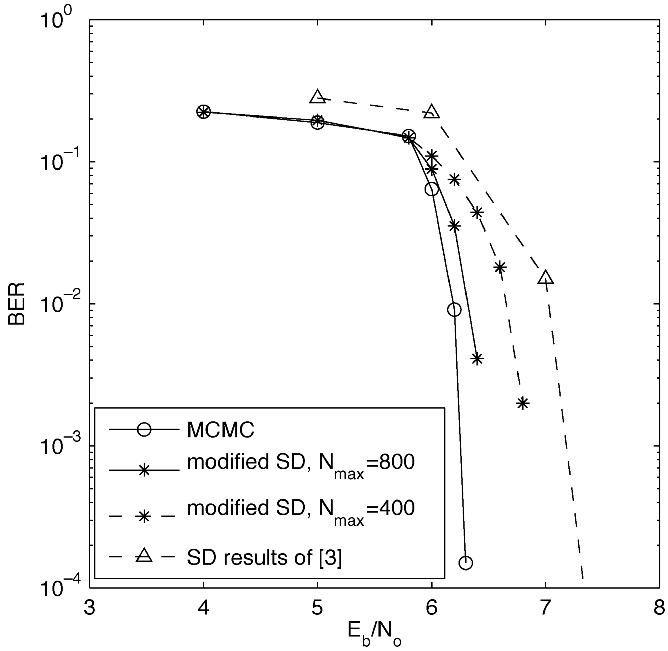

Fig. 4. BER results of the SD, modified SD, MCMC, and exact max-log SD detectors for a MIMO system with eight transmit and eight receive antennas.

TABLE I
COMPLEXITY STUDY OF SD AND MCMC DETECTORS

	SNR (dB)	Channel size	L, N_S or $N_{\max}$, $N_{\min}$	No. of Operations
MCMC	5.6	4×4	10, 10	1.32×10^8
SD	5.6	4×4	100, 10	1.16×10^8
MCMC	6.5	8×8	20, 20	9.72×10^8
SD	6.5	8×8	400, 10	1.40×10^{10}
SD	6.5	8×8	800, 10	1.69×10^{10}

of SD is a direct consequence of the fact that the SD tree search algorithm often encounters many terminal nodes before finding a sample of $\mathbf{d}$, and it grows exponentially with the size of $\mathbf{d}$. The number of terminal nodes increases further in the particular case of iterative detectors, where many samples of $\mathbf{d}$ have to be collected.

Table I presents the number of multiplication and addition operations per data block, counted while running our simulation programs, at the indicated SNR values. Measuring the complexity by these numbers, SD and MCMC detectors exhibit similar complexity in the case of a 4×4 MIMO channel. However, because of the exponential growth of the complexity of SD with the size of $\mathbf{d}$, in the case of an 8×8 channel, the complexity difference between MCMC and SD grows to over an order of magnitude.

VIII. CONCLUSION

In this letter, we explored the similarities and the differences of SD and the statistical methods that work based on MCMC simulation in soft MIMO detectors. It was noted that both methods start with construction of a candidate list of more likely samples of the transmitted symbols. While SD takes a deterministic approach (a tree search), MCMC detector obtains the samples through a statistical procedure. The candidate list is then used for computation of the LLR values of the information bits. We found that the way the candidate list is handled by the reported SD algorithms [3], [4] is different from what has been suggested in a recently proposed MCMC detector [7], [8]. By applying the method of [7] and [8] to SD, we proposed a new detector and called it modified SD. Moreover, we proposed an exact max-log SD detector that may be used as a benchmark for evaluation of the MCMC and SD detectors. Computer simulations revealed that while an MCMC detector with moderate complexity can approach the performance of the exact max-log SD detector, SD, even after the modification proposed in this letter, falls short in performance. We also briefly discussed the complexity of SD and MCMC detection and found that in low SNR regime, the latter may offer a significantly lower computational complexity.

REFERENCES

[1] X. Wang and H. V. Poor, "Iterative (Turbo) soft interference cancellation and decoding for coded CDMA," *IEEE Trans. Commun.*, vol. 47, no. 7, pp. 1046–1061, Jul. 1999.

[2] H. E. Gamal and A. R. Hammons, Jr., "A new approach to layered space-time coding and signal processing," *IEEE Trans. Inf. Theory*, vol. 47, no. 6, pp. 2321–2334, Sep. 2001.

[3] B. M. Hochwald and S. ten Brink, "Achieving near-capacity on a multiple-antenna channel," *IEEE Trans. Commun.*, vol. 51, no. 3, pp. 389–399, Mar. 2003.

[4] H. Vikalo and B. Hassibi, "Modified Fincke–Pohst algorithm for low-complexity iterative decoding over multiple antenna channels," in *Proc. IEEE Int. Symp. Inf. Theory*, 2002, p. 390.

[5] C. P. Robert and G. Casella, *Monte Carlo Statistical Methods*. New York: Springer-Verlag, 1999.

[6] O. Häggström, *Finite Markov Chains and Algorithmic Applications*. Cambridge, U.K.: Cambridge Univ. Press, 2002.

[7] Z. Shi, H. Zhu, and B. Farhang-Boroujeny, "Markov chain Monte Carlo techniques in iterative detectors: a novel approach based on Monte Carlo integration," in *Proc. Globecom*, Dallas, TX, Nov. 29 to Dec. 2, 2004, pp. 325–329.

[8] B. Farhang-Boroujeny, H. Zhu, and Z. Shi, "Markov Chain Monte Carlo Algorithms for CDMA and MIMO Communication Systems," *IEEE Trans. Signal Process.*, to be published.

[9] P. Robertson, E. Villebrun, and P. Hoeher, "A comparison of optimal and sub-optimal MAP decoding algorithms operating in the log domain," in *Proc. IEEE Int. Conf. Commun.*, vol. 2, Seattle, WA, Jun. 18–22, 1995, pp. 1009–1013.

[10] D. Pham, K. R. Pattipati, P. K. Willett, and J. Luo, "An improved complex sphere decoder for V-BLAST systems," *IEEE Signal Process. Lett.*, vol. 11, no. 3, pp. 748–751, Sep. 2004.

[11] B. Hassibi and H. Vikalo, "On the sphere decoding algorithm I. Expected complexity," *IEEE Trans. Signal Process.*, pt. 1, vol. 53, no. 8, pp. 2806–2818, Aug. 2005.

[12] J. Luo, K. R. Pattipati, P. Willett, and G. G. Levchuk, "Fast optimal and suboptimal any-time algorithms for CDMA multiuser detection based on branch and bound," *IEEE Trans. Commun.*, vol. 52, no. 4, pp. 632–642, Apr. 2004.