CRITICAL POINTS OF THREE–DIMENSIONAL BOOTSTRAP PERCOLATION–LIKE CELLULAR AUTOMATA

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Abstract. This paper deals with the critical point of three-dimensional bootstrap percolation-like cellular automata. Some general sufficient or necessary conditions for $p_c = 0$ are obtained. In the case of $p_c > 0$, some explicit upper and lower bounds are provided in terms of the critical value of oriented site percolation.

1. Introduction.

This section consists of two parts. In part 1, we introduce some models which will be dealt with in the paper. In part 2, we review some related known results.

1. Models. We recall the cellular automaton models introduced by Schonmann [1].

The Lattice. Consider the lattice $\mathbb{Z}^d$ with the ℓ_1-norm $\|x\| = \sum_{i=1}^d |x_i|$. Denote by $\mathcal{N}_x (x \in \mathbb{Z}^d)$ the nearest neighbors of x: $\mathcal{N}_x = \{y \in \mathbb{Z}^d; \|y - x\| = 1\}$. Set $\mathcal{N} = \mathcal{N}_0$. The models studied in the paper are specified case by case by a class $\mathcal{D}$ of some subsets of $\mathcal{N}$. Throughout the paper, we assume that $\mathcal{D}$ possesses the monotonicity: if $A \in \mathcal{D}$ and $A \subset B$, then $B \in \mathcal{D}$.

The Systems. Consider $\eta_t : \mathbb{Z}^d \to \{0, 1\}$ with discrete time $t = 0, 1, 2, \cdots$. We say that the site x is **empty** (resp. **occupied**) at time t, if $\eta_t(x) = 0$ (resp. 1). The system always starts (at $t = 0$) from a translation invariant product random field; that is, the random variables $\eta_0(x)$, $x \in \mathbb{Z}^d$ are i.i.d. with distribution $\mathbb{P}(\eta_0(x) = 1) = p$. We call $p \in [0, 1]$ the **initial density**. The system then evolves according to the following sort of deterministic rules:

(1) If $\eta_t(x) = 1$, then $\eta_{t+1}(x) = 1$ (1's are stable).
(2) If $\eta_t(x) = 0$ and $\{y : \eta_t(y) = 1\} \cap \mathcal{N}_x \in \mathcal{D}_x$, then $\eta_{t+1}(x) = 1$, otherwise $\eta_{t+1}(x) = 0$.

The standard basis in $\mathbb{Z}^d$ are denoted by $e_1 = (1, 0, 0, \cdots, 0)$, $\cdots$, $e_d = (0, 0, 0, \cdots, 1)$. Next, denote by $|A|$ the cardinality of set A.

Examples. (1) **Bootstrap percolation**. Take $\ell \in \{0, \cdots, 2d\}$ and $\mathcal{D} = \{A \subset \mathcal{N} : |A| \geq \ell\}$. In words, a 0 becomes 1 if at least ℓ of its neighbors are 1's.

(2) **The basic model**. It is the particular case of bootstrap percolation with $\ell = d$.

(3) **The modified basic model**. $\mathcal{D} = \{A \subset \mathcal{N} : A \cap \{-e_i, +e_i\} \neq \emptyset$ for $i = 1, \cdots, d\}$. In this model a 0 becomes a 1 if in each one of the d coordinates directions it has at least one neighbor which is a 1.

(4) **Oriented models**. Take $(a_1, \cdots, a_d) \in \{-1, +1\}^d$. For each one of the 2^d choices we have one of the oriented models defined by $\mathcal{D} = \{A \subset \mathcal{N} : \{a_1 e_1, a_2 e_2, \cdots, a_d e_d\} \subset A\}$. In the case that $a_i = +1$ for $i = 1, \cdots, d$, the model is called the **basic oriented model**.

Given two models specified by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, we say that **the latter dominates the former** if $\mathcal{D}_1 \subset \mathcal{D}_2$. Informally, if a 0 becomes a 1 in the former, the same occurs in the latter. The following statements are clearly true. The bootstrap percolation

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model with $\ell = \ell_1$, dominates the one with $\ell = \ell_2$ if $\ell_1 \le \ell_2$. The basic model dominates the modified basic model which then dominates all the oriented models.

Problems. Endow $\{0,1\}^{\mathbb{Z}^d}$ with the product topology and denote by Σ its Borel σ-algebra. Let $\mathcal{P}$ be the set of probability measures on $(\{0,1\}^{\mathbb{Z}^d}, \Sigma)$. On $\mathcal{P}$ we define the following partial order. If $\mu, \nu \in \mathcal{U}$, we say that ν dominates μ and write $\mu \le \nu$ if $\int f(\eta) d\mu(\eta) \le \int f(\eta) d\nu(\eta)$ for every continuous nondecreasing function $f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$. Here, the ordinary partial order in used on $\{0,1\}^{\mathbb{Z}^d}$: $\eta \le \eta'$ iff $\eta(x) \le \eta'(x)$ for all $x \in \mathbb{Z}^d$. Let $\mu_0^p =$ (translation invariant product measure with density p) be the initial distribution on $\{0,1\}^{\mathbb{Z}^d}$ and let μ_t^p be the corresponding distribution of the process $(\eta_t : t \ge 0)$. at time t. Since the $1'$s are stable we have $\mu_0^p \le \mu_1^p \le \mu_2^p \le \cdots$. Since $\{0,1\}^{\mathbb{Z}^d}$ is compact and so is $\mathcal{P}$, it follows that μ_t^p converges weakly to a probability distribution $\mu^p \in \mathcal{P}$. Clearly, μ^p is translation invariant. The asymptotic density is defined by $\rho(p) = \mu^p\{\eta : \eta(0) = 1\}$. By attractiveness, we have $\mu^{p_1} \le \mu^{p_2}$ whenever $p_1 \le p_2$. In particular $\rho(p_1) \le \rho(p_2)$.

We are mainly interested in the following problems.

1) When is it the case that $\rho(p) = 1$? Clearly $\rho(0) = 0$ and $\rho(1) = 1$. From the monotonicity of ρ, it is natural to define $p_c = \inf\{p \in [0,1] : \rho(p) = 1\}$.

2) Define the random time $T = \inf\{t \ge 0 : \eta_t(0) = 1\}$. The question is whether there exists constants $\gamma, C \in (0, \infty)$ such that $\mathbb{P}(T > t) \le Ce^{-\gamma t}$. We may define for every p, $\gamma(p) = \sup\{\gamma \ge 0 :$ there exists a $C < \infty$ such that $\mathbb{P}(T > t) \le Ce^{-\gamma t}\}$. Since γ is monotonic, it is natural to define another critical point $\pi_c = \inf\{p \in [0,1] : \gamma(p) > 0\}$. Clearly, we have $p_c \le \pi_c$.

2. Some Known Results.

It is not difficult to show that if a model does not dominate any oriented model, then it has $p_c = \pi_c = 1$[1]. On the other hand, for models that dominate some oriented models, $p_c \le \pi_c < 1$. Thus when we consider the critical values p_c and π_c, we are only interested in those models which dominate some oriented models.

In order to state the further results, we need the close related **oriented site percolation model**. Again, each site $x \in \mathbb{Z}^d$ is occupied independently with probability p. We say that $(x_1, x_2, \cdots, x_n)$ is an **oriented path** in $\mathbb{Z}^d$ with length n if $x_i \in \mathbb{Z}^d, i = 1, \cdots, n$ and either $n = 1$ or $x_{i+1} - x_i \in \{e_1, \cdots, e_d\}, i = 1, \cdots, n-1$. Define

$$p_c^* = \inf\{p \in [0,1] : \mathbb{P}[\text{there is an infinite oriented path}] > 0\}.$$

Then it is well known that $p_c^* = 1$ when $d = 1$ and $0 < p_c^* < 1$ when $d \ge 2$. Certainly, all the critical values p_c, π_c and p_c^* depend on the dimension d. So sometimes, we write $p_c^*(d)$ instead of p_c^* if it is necessary.

The following result is taken from [1; Proposition 4.2 and Theorem 3.1]):

Theorem 1.1. (1) *For the oriented models, we have* $0 < p_c = \pi_c = 1 - p_c^* < 1$.
(2) *For the modified basic model, we have* $p_c = \pi_c = 0$ *in all dimensions.*

Schonmann[2] presented a classification of two-dimensional models.

Theorem 1.2. *Let* $d = 2$. *Assume that* $\{e_1, e_2\} \in \mathcal{D}$.

(1) *If at least one of the sets* $\{-e_1, e_2\}, \{e_1, -e_2\}$ *is in* $\mathcal{D}$, *then* $p_c = 0$.
(2) *If neither of the sets in* (1) *belongs to* $\mathcal{D}$, *then we have* $1 - {p_c^*}^{1/4} \le p_c \le 1 - p_c^*$.

Except the results listed above, our knowledge about the critical values p_c and π_c are far away to be complete. Indeed, the main aim of the paper is to present some sufficient or necessary conditions for $p_c = 0$ (Theorem 2.1 and Theorem 2.4) in three-dimensional situation. Some analogs of Theorem 1.2 (2) are also presented. Next, one can also study the critical value $p_c = p_c(d, L)$ replacing $\mathbb{Z}^d$ with d-dimensional hypercubic lattice with large linear size L. For instance, Aizenman and Lebowitz[4] showed that the finite-size scaling of the bootstrap percolation model with $\ell = 2$, $d = 2$ is $p_c \sim O(1/\log L)$. For the bootstrap percolation model with $\ell = 2, d = 3$, van Enter, Adler and Duarte[5] showed that the corresponding scaling is $p_c \sim O(1/\log(\log L))$. Furthermore, they showed[6] that the finite scaling of the Schonmann's two-dimensional directed model in Theorem 1.2 satisfies $p_c/(\log p_c)^2 < O(1/\log L)$. Our Theorem 2.3 below says that for the model given in Theorem 2.1, we have $p_c \sim O(1/\log(\log L))$.

2. Critical Points for Three–dimensional Models

We introduce the **characteristic set** $\mathcal{D}^*$:

$$\mathcal{D}^* = \{A \in \mathcal{D} : \text{ for every } B \subset A \text{ and } B \neq A, \text{ we have } B \notin \mathcal{D}\}.$$

Clearly, both $\mathcal{D}$ and $\mathcal{D}^*$ can be used to describe (or specify) the growth rules of the model. Actually, each representation has its own advantages, we will use both of them according to our convenience.

Theorem 2.1. *Let $\mathcal{D}_0^*$ denote the characteristic set of the three-dimensional modified basic model, that is $\mathcal{D}_0^* = \{\{a_1e_1, a_2e_2, a_3e_3\},\ a_1, a_2, a_3 = 1, -1\}$. Then for every model obtained by subtracting an arbitrary element in $\mathcal{D}_0^*$, we have $p_c = 0$.*

To prove Theorem 2.1, we need some notations.

(1) We say that a finite set $\Gamma \subset \mathbb{Z}^d$ is **internally spanned** by the configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ if starting from the configuration η^Γ: $\eta^\Gamma(x) = \eta(x)$ if $x \in \Gamma$ and $= 0$ otherwise, and letting the system evolve according to the d-dimensional dynamics restricted to Γ, Γ will eventually become completely occupied. Let $Q_N^d = \{x \in \mathbb{Z}^d : |x_i| \leq N, i = 1, \cdots, d\}$. Define

$$R^d(N,p) = \mathbb{P}[Q_N^d \text{ is internally spanned by a random configuration chosen according to a product measure with density } p].$$

For simplicity, in what follows, we write $R(N,p) = R^2(N,p)$.

(2) We introduce the following sets:

$$A(i,k) = \{(x_1,x_2,x_3) \in \mathbb{Z}^3 : \ x_1 = i,\ 0 \leq x_2,\ x_3 \leq k\}$$
$$B(i_1,i_2,k) = \{(x_1,x_2,x_3) \in \mathbb{Z}^3 : i_1 \leq x_1 \leq i_2,\ 0 \leq x_2 \leq i_2 - i_1,\ x_3 = k\}$$
$$C(i_1,i_2,k) = \{(x_1,x_2,x_3) \in \mathbb{Z}^3 : i_1 \leq x_1 \leq i_2,\ x_2 = k,\ 0 \leq x_3 \leq i_2 - i_1 + 1\}$$
$$D(i_1,i_2,k) = \{(x_1,x_2,x_3) \in \mathbb{Z}^3 : i_1 \leq x_1 \leq i_2,\ x_2 = k,\ 0 \leq x_3 \leq i_2 - i_1\},\ k \in \mathbb{N}.$$

(3) Define a sequence $\{m_k\}$ as follows: $m_1 = -1$, $m_{k+1} = m_k - k$, $k \geq 1$. That is, $m_k = -k(k-1)/2 - 1$.

Proof of Theorem 2.1. Because of the geometric symmetry, we need only to consider the case that $\mathcal{D}^* = \mathcal{D}_0^* \setminus \{-e_1, e_2, e_3\}$.

a) First, we prove that if $A(m_k, k)$ is (completely) occupied at time $t = t_0$, then

$$\text{Prob.}[A(m_{k+1}, k+1) \text{ will become occupied at some time } t \geq t_0] \geq [1-(1-p)^k] R(k,p)^{k+2}.$$

Here we write "Prob." rather than "$\mathbb{P}$" since the probability is conditional.

According to our assumption, $\{\{e_1, a_2e_2, a_3e_3\},\ a_2,\ a_3 = 1, -1\} \subset \mathcal{D}^*$. Thus, if $A(m_k, k)$ is occupied at $t = t_0$, then

$$\text{Prob.}[A(m_k - 1, k) \text{ will become occupied at some } t = t_1 \geq t_0] \geq R(k,p).$$

Similarly, we have

$$\text{Prob.}[\text{All } A(m_k - i, k),\ 1 \leq i \leq k \text{ will become occupied at some time } t = t_2] \geq R(k,p)^k.$$

Next, because $\{\{a_1e_1, a_2e_2, -e_3\},\ a_1, a_2 = 1, -1\} \subset \mathcal{D}^*$, if all $A(m_k - i, k), 1 \le i \le k$ are occupied at $t = t_2$, then at the same time $B(m_{k+1}, m_k, k)$ is also occupied and

$$\text{Prob.}[B(m_{k+1}, m_k, k+1) \text{ will be occupied at some } t = t_3 \ge t_2] \ge R(k,p).$$

Assume that $A(m_k - i, k)$, $1 \le i \le k$ and $B(m_{k+1}, m_k, k+1)$ are all occupied at time $t = t_3$. Then $C(m_{k+1}, m_k, k)$ is also occupied at the same time. We now claim that

$$\text{Prob.}[C(m_{k+1}, m_k, k+1) \text{ will become occupied at some } t = t_4 \ge t_3] \ge [1-(1-p)^k]R(k,p).$$

Actually, if at some time t, $D(m_{k+1}, m_k, k+1)$ is occupied and meantime there is at least one occupied site on the segment $C(m_{k+1}, m_k, k+1) \setminus D(m_{k+1}, m_k, k+1)$, then $C(m_{k+1}, m_k, k+1)$ must be occupied at some $t = t_4$. Here we also have to use the facts that $\{\{a_1e_1, -e_2, a_3e_3\},\ a_1, a_3 = 1, -1\} \subset \mathcal{D}^*$ and at $t = t_3$, $C(m_{k+1}, m_k, k)$ is occupied. So at $t = t_4$, $A(m_{k+1}, k+1)$ becomes occupied. ¿From the above statements, our desired assertion easily follows:

$$\begin{aligned}
&\text{Prob.}[A(m_{k+1}, k+1) \text{ will be occupied at some } t \ge t_0] \\
&\quad \ge R(k,p)^k R(k,p)[1-(1-p)^k]R(k,p) \\
&\quad = [1-(1-p)^k]R(k,p)^{k+2}.
\end{aligned}$$

b) Let

$$\alpha(p) = \mathbb{P}[\text{ all } A(m_k, k), k \ge 1 \text{ will become occupied}].$$

We now prove that $\alpha(p) > 0$. From a), we have

$$\alpha(p) \ge p^4 \Pi_{k=1}^{\infty}(1-(1-p)^k)R(k,p)^{k+2},$$

where $p^4 = \mathbb{P}[A(-1,1) \text{ is occupied at } t = 0]$.

It is known that for two-dimensional modified basic model, there exists $\gamma(p) > 0$ and $C(p) < \infty$ such that

$$1 - R(k,p) \le C(p)\exp[-\gamma(p)k]$$

for every $p > 0$ and $k \in \mathbb{N}$. From this, it is not difficult to show that $\log \alpha(p) > -\infty$, which implies $\alpha(p) > 0$.

c) The origin is said to be a **good site** in the configuration η if the system starts from η, all the sets $A(m_k, k)$, $k = 1, 2, \cdots$ will become occupied eventually. This implies the negative half line $(-k, 0, 0), k = 1, 2, \cdots$ will be finally occupied. Put $B = \{\eta : \text{the origin is a good site in } \eta\}$. Define the shift operator $T^i : T^i(\eta)(x) = \eta(x + ie_1)$, $x \in \mathbb{Z}^3$. By the ergodicity of product measure, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} 1_B(T^i\eta) \longrightarrow \mu(B) \ge \alpha(p) > 0.$$

This means that with probability one, there exists an $i_0 > 0$ such that $T^{i_0}(\eta) \in B$. It follows that the half x-axis $(x_1, 0, 0)$, $x_1 \le i_0 - 1$ will be occupied. In particular, the origin will be occupied. This completes the proof of $p_c = 0$. □

Theorem 2.3. *For the models introduced in Theorem 2.1, we have the threshold scaling* $O\big(1/\log(\log L)\big)$ *for a finite system of size* L.

Proof. Here we modify the definition of a good site given in the proof of Theorem 2.1. Given p, the origin is called a **good site** in the configuration η if the sets $A(m_k, k)$, $k = 1, 2, \cdots, [e^{1/p}]$ are occupied simultaneously in η and the system starts from η, all the sets

$A(m_k, k)$, $k = [e^{1/p}]+1, [e^{1/p}]+2, \cdots$ will become occupied eventually. Similarly, we define $x \in \mathbb{Z}^d$ to be a good site if the origin is a good site in the configuration $\theta_{-x}\eta$. Also from the proof of Theorem 2.1, we see that the probability of a particular point is a good site in the present sense is at least $\alpha(p) = \Pi_{k=1}^{[e^{1/p}]} p^{(k+1)^2} \Pi_{k=[e^{1/p}]}^{\infty} [1-(1-p)^k] R(k,p)^{k+2}$. Hence,

$$\log\alpha(p) = (\log p)\Big(\sum_{k=1}^{[e^{1/p}]}(k+1)^2\Big) + \sum_{k=[e^{1/p}]}^{\infty}\log[1-(1-p)^k] + \sum_{k=[e^{1/p}]}^{\infty}(k+2)\log R(k,p).$$

Now we discuss the asymptotics of these three terms respectively as $p \to 0$.

Firstly,

$$(\log p)\Big(\sum_{k=1}^{[e^{1/p}]}(k+1)^2\Big) = (\log p)\frac{([e^{1/p}]+1)([e^{1/p}]+2)(2[e^{1/p}]+3)-1}{6} \sim O\Big((\log p)e^{3/p}\Big).$$

Secondly, $\sum_{k=[e^{1/p}]}^{\infty}\log(1-(1-p)^k) \sim \sum_{k=[e^{1/p}]}^{\infty}(1-p)^k \sim O(1/p)$. Thirdly, by [3; Theorem 2], we have $\dfrac{1-R(N,p)}{2(2N+1)(1-p)^{2N+1}} \to 1$ as $N \to \infty$. Therefore

$$\begin{aligned}
\textstyle\sum_{k=[e^{1/p}]}^{\infty}(k+2)\log R(k,p) &= \textstyle\sum_{k=[e^{1/p}]}^{\infty}(k+2)\log[1-(1-R(k,p))] \\
&= O\Big(\textstyle\sum_{k=[e^{1/p}]}^{\infty}(k+2)[1-R(k,p)]\Big) \\
&= O\Big(\textstyle\sum_{k=[e^{1/p}]}^{\infty}(k+2)2(2k+1)(1-p)^{2k+1}\Big) \\
&= O\Big(\textstyle\sum_{k=2}^{\infty}k(k-1)[(1-p)^2]^{k-2}\Big) \\
&= O\big(p^{-3}\big).
\end{aligned} \tag{2.1}$$

The last step holds because if we define $f(r) = \sum_{k=1}^{\infty} r^k = -1 + 1/(1-r)$, then $f'(r) = \sum_{k=1}^{\infty} kr^{k-1} = 1/(1-r)^2$ and $f''(r) = \sum_{k=2}^{\infty} k(k-1)r^{k-2} = 2/(1-r)^3$. Let $r = (1-p)^2$, then we easily get (2.1).

Thus the first term gives the asymptotics

$$\log\alpha(p) \sim O\Big((\log p)e^{3/p}\Big) \text{ as } p \to 0. \tag{2.2}$$

Now we invert (2.2) to obtain the system size $L^3 \sim 1/\alpha(p)$ that is needed to contain at least one critical wedge in the proof of Theorem 2.1. This gives us $p \sim O\big(1/\log(\log L)\big)$. $\square$

In the remaining part of this section we shall give some necessary conditions for $p_c = 0$. As we mentioned before, we need only to study the models which dominate three-dimensional oriented basic model, namely $\{e_1, e_2, e_3\} \in \mathcal{D}$. It easily follows for these models $p_c \leq 1 - p_c^*(3)$.

Theorem 2.4. *Let $\{e_1, e_2, e_3\} \in \mathcal{D}$. For a model with $p_c = 0$, it is necessary that all of the following three conditions holds.*

(1) *At least one of the sets $\{e_1, e_2, -e_3\}, \{e_1, -e_2, -e_3\}, \{-e_1, e_2, e_3\}, \{-e_1, -e_2, e_3\}$ belongs to $\mathcal{D}$.*
(2) *At least one of the sets $\{e_1, e_2, -e_3\}, \{e_1, -e_2, e_3\}, \{-e_1, e_2, -e_3\}, \{-e_1, -e_2, e_3\}$ belongs to $\mathcal{D}$.*
(3) *At least one of the sets $\{e_1, -e_2, e_3\}, \{e_1, -e_2, -e_3\}, \{-e_1, e_2, e_3\}, \{-e_1, e_2, -e_3\}$ belongs to $\mathcal{D}$.*

For those models which do not satisfy any of the above conditions, we have

$$1 - p_c^*(2)^{1/8} \le p_c \le 1 - p_c^*(3). \tag{2.3}$$

Proof. Because the geometric symmetry of e_1, e_2 and e_3, we need only to consider the model which does not satisfy (1) and show that (2.3) holds. First, we introduce some notation. Define $Q_0 = \{x \in Z^3,\ x_i = 0, 1,\ i = 1, 2, 3\}$. For every $k = (k_1, k_2, k_3) \in Z^3$, let $Q_k = Q(k_1, k_2, k_3) = Q_0 + 2k$.

Consider a new lattice in $\mathbb{Z}^2$ with coordinates k_1, k_3 and declare the site (k_1, k_3) of this new lattice to be vacant at time t if and only if all of the eight sites in Q_k are vacant at the same time in the original lattice, where $k = (k_1, 0, k_3)$. Assume that at time $t = 0$, double-oriented percolation of vacant sites occurs in the new lattice. This means that there is a doubly infinite chain of sites, $\cdots, z_{-2}, z_{-1}, z_0, z_1, z_2, \cdots$ in the configuration η such that $z_0 = 0$, $\eta(z_i) = 0$ and $z_{i+1} - z_i = (1, 0)$ or $(0, 1)$. Then we show that under our assumption, each site of this infinite vacant chain will remain vacant at any later time. Equivalently, in our original lattice, all the cubes Q_k corresponding to the infinite chain in the new lattice will remain vacant at all time as well. The proof goes as follows: At $t = 0$, assume that (k_1, k_3) belongs to the infinite vacant chain in the new lattice. Then, according to our definition, the cube Q_k, where $k = (k_1, 0, k_3)$, is vacant at time $t = 0$. Moreover, based on the structure of the neighbor cubes of Q_k, one of the following four cases should happen:

(1) $Q(k_1 - 1, 0, k_3)$ and $Q(k_1 + 1, 0, k_3)$ are vacant.
(2) $Q(k_1, 0, k_3 - 1)$ and $Q(k_1, 0, k_3 + 1)$ are vacant.
(3) $Q(k_1, 0, k_3 - 1)$ and $Q(k_1 + 1, 0, k_3)$ are vacant.
(4) $Q(k_1 - 1, 0, k_3)$ and $Q(k_1, 0, k_3 + 1)$ are vacant.

In case (1), all the eight sites in $Q(k_1, 0, k_3)$ remain vacant at $t = 1$ because none of the sets $\{a_1e_2, a_2e_3\}$, $a_1,\ a_2 = 1, -1$ belongs to $\mathcal{D}$. This follows from the property of $\mathcal{D}$ mentioned at the beginning of the paper. The same occurs in case (2), since none of the sets (a_1e_1, a_2e_2), $a_1,\ a_2 = 1,\ -1$ belongs to $\mathcal{D}$. In case (3), in $Q(k_1, 0, k_3)$, the two sites $(2k_1, 0, 2k_3 + 1)$, $(2k_1, 1, 2k_3 + 1)$ remain vacant at $t = 1$ because neither of the two sets $\{-e_1, -e_2, e_3\}$, $\{-e_1, e_2, e_3\}$ belongs to $\mathcal{D}$. Other sites remain vacant at $t = 1$ because none of the sets $\{a_1e_2, a_2e_3\}$, $\{a_1e_1, a_2e_2\}$, $a_1,\ a_2 = 1,\ -1$ belongs to $\mathcal{D}$. Similarly, in case (4), consider $Q(k_1, 0, k_3)$, the two sites $(2k_1 + 1, 0, 2k_3)$, $(2k_1 + 1, 1, 2k_3)$ remain vacant at $t = 1$ because neither of the two sets $\{e_1, -e_2, -e_3\}$, $\{e_1, e_2, -e_3\}$ belongs to $\mathcal{D}$. Other sites also remain vacant at $t = 1$ because none of the sets $\{a_1e_2, a_2e_3\}$, $\{a_1e_1, a_2e_2\}$, $a_1,\ a_2 = 1,\ -1$ belongs to $\mathcal{D}$.

Thus, we have seen that for every (k_1, k_3) belongs to the infinite vacant chain in the new lattice at $t = 0$, the corresponding cubic $Q(k_1, 0, k_3)$ in the original lattice will remain to be vacant at $t = 1$. Induction on t leads to the conclusion that if doubly-oriented percolation occurs in our new lattice, it will remain empty later. In other words, if p satisfies $(1 - p)^8 \ge p_c^*(2)$, then we will have $p \le p_c$. This implies our assertion. □

The next result follows immediately from Theorem 2.4 but it is not a consequence of Theorem 2.6 below.

Corollary 2.5. *If $\mathcal{D}^* = \{\{e_1, e_2, e_3\}, \{e_1, -e_2, e_3\}, \{-e_1, e_2, -e_3\}, \{-e_1, -e_2, -e_3\}\}$, then we have $1 - p_c^*(2)^{1/8} \le p_c \le 1 - p_c^*(3)$.*

Theorem 2.6. *A necessary condition for $p_c = 0$ is as follows: for every pair of i, j, $(1 \le i \ne j \le 3)$, there exists an $A \in \mathcal{D}$ such that $A \cap \{\pm e_i, \pm e_j\} = \{-e_i, e_j\}$ or $\{e_i, -e_j\}$. For those models which do not satisfy the above condition, we have $1 - p_c^*(2)^{1/4} \le p_c \le 1 - p_c^*(3)$.*

Proof. Again, because of geometric symmetry, we need only to consider the case of $i = 1$ and $j = 2$. Our method is to compare our model with a two-dimensional bootstrap percolation model which is intuitively the projection of our model on $e_1 \times e_2$-plane. We denote by $\mathcal{D}'$ this new two-dimensional model, where

$$\mathcal{D}' = \{A \subset \{\pm e_1, \pm e_2\} : \text{ there exists } B \in \mathcal{D} \text{ such that } A = B \cap \{\pm e_1, \pm e_2\}\}.$$

Now assume that neither of the two sets: $\{e_1, -e_2\}, \{-e_1, e_2\}$ belongs to $\mathcal{D}'$. It then follows from Theorem 1.2 that this two-dimensional model has the critical value $p_c \geq 1 - p_c^*(2)^{1/4}$. Since it dominates our original model, the assertion of the theorem easily follows. $\square$

Corollary 2.7. *If* $\mathcal{D}^* = \{\{e_1, e_2, e_3\}, \{e_1, -e_2, e_3\}, \{-e_1, e_2, -e_3\}\}$, *then we have* $1 - p_c^*(2)^{1/4} \leq p_c \leq 1 - p_c^*(3)$.

Proof. Since there does not exist a set $A \in \mathcal{D}$ such that $A \cap \{\pm e_1, \pm e_3\} = \{-e_1, e_3\}$ or $\{e_1, -e_3\}$, the corollary easily follows from Theorem 2.6. $\square$

Theorem 2.8 (1/4 oriented model). *For each of the models:*

(1) $\mathcal{D}^* = \{\{e_1, e_2, e_3\}, \{e_1, e_2, -e_3\}\}$
(2) $\mathcal{D}^* = \{\{e_1, e_2, e_3\}, \{-e_1, e_2, e_3\}\}$
(3) $\mathcal{D}^* = \{\{e_1, e_2, e_3\}, \{e_1, -e_2, e_3\}\}$

we have $1 - p_c^*(2) \leq p_c \leq 1 - p_c^*(3)$.

Proof. Here we prove the statement (1) only since the proof for the other models is similar. Actually, our model is dominated by the basic oriented model in the $e_1 \times e_2$-plane, whose critical value p_c is $1 - p_c^*(2)$ by Theorem 1.1 (1). $\square$

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