ECE 7520, Spring 2011

Solution to Homework #1

1. **Entropy of functions of a random variable.** Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps $(a), (b), (c)$ and (d):

$$\begin{aligned} H(X, g(X)) &\overset{(a)}{=} H(X) + H(g(X)|X) \\ &\overset{(b)}{=} H(X) \\ H(X, g(X)) &\overset{(c)}{=} H(g(X)) + H(X|g(X)) \\ &\overset{(d)}{\geq} H(g(X)) \end{aligned}$$

Hence $H(g(X)) \leq H(X)$.

Solution:

(a) By the chain rule for entropies.

(b) Given X, $g(X)$ has a fixed value. Hence

$$H(g(X)|X) = \sum_x p(x) H(g(X)|X = x) = \sum_x 0 = 0.$$

(c) By the chain rule for entropies.

(d) Follows because the (conditional) entropy of a discrete random variable is nonnegative, i.e., $H(X|g(X)) \geq 0$, with equality iff $g(X)$ is a one-to-one mapping.

2. **A measure of correlation.** Let X_1 and X_2 be identically distributed, but not necessarily independent. Let

$$\rho = 1 - \frac{H(X_1|X_2)}{H(X_1)}.$$

(a) Show $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.

(b) Show $0 \leq \rho \leq 1$.

(c) When is $\rho = 0$?

(d) When is $\rho = 1$?

Solution:

(a)$\rho = \frac{H(X_1) - H(X_1|X_2)}{H(X_1)} = \frac{I(X_1;X_2)}{H(X_1)}$

(b) $0 \leq \rho \leq 1$ follows easily because $0 \leq H(X_1|X_2) \leq H(X_1)$.

(c) $\rho = 0$ iff $I(X_1; X_2) = 0$, i.e., X_1 and X_2 are independent.

(d) $\rho = 1$ iff $H(X_1|X_2) = 0$, i.e., X_1 is a function of X_2.

3. **Example of joint entropy.** Let $p(x, y)$ be given by

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find

(a) $H(X)$, $H(Y)$.

(b) $H(X|Y)$, $H(Y|X)$.

(c) $H(X,Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X;Y)$.

(f) Draw a Venn diagram for the quantities in (a) through (e).

Solution:

(a) $H(X) = \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 = \log 3 - \frac{2}{3} = 0.918$ bits $= H(Y)$.

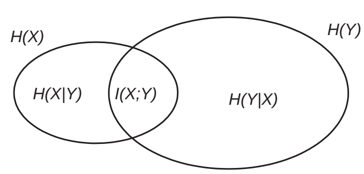
(b) $H(X|Y) = \frac{1}{3}H(X|Y=0) + \frac{2}{3}H(X|Y=1) = \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 1 = \frac{2}{3} = 0.667$ bits $= H(Y|X)$.

(c) $H(X,Y) = 3 \times \frac{1}{3}\log 3 = \log 3 = 1.585$ bits. Alternatively, $H(X,Y) = H(X) + H(Y|X) =$ 1.585 bits.

(d) $H(Y) - H(Y|X) = \log 3 - \frac{2}{3} - \frac{2}{3} = \log 3 - \frac{4}{3} = 0.251$ bits.

(e) $I(X;Y) = H(Y) - H(Y|X) = 0.251$ bits.

(f)



4. **Mixing increases entropy.** Show that the entropy of the probability distribution

$$(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$$

is less than the entropy of the distribution

$$(p_1, \ldots, \frac{p_i + p_j}{2}, \ldots, \frac{p_i + p_j}{2}, \ldots, p_m).$$

(hint: $\log_e x \leq x - 1$ for $x > 0$.)

Solution:

Let $P_1 = (p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$ and $P_2 = (p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_m)$.

$$\begin{aligned} H_e(P_1) - H_e(P_2) &= -p_i \ln p_i - p_j \ln p_j + 2\frac{p_i + p_j}{2} \ln \frac{p_i + p_j}{2} \\ &= p_i \ln \frac{p_i + p_j}{2p_i} + p_j \ln \frac{p_i + p_j}{2p_j} \\ &\leq p_i \left(\frac{p_i + p_j}{2p_i} - 1 \right) + p_j \left(\frac{p_i + p_j}{2p_j} - 1 \right) \\ &= 0. \end{aligned}$$

5. **Relative entropy is not symmetric.** Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions $p(x)$ and $q(x)$ on this random variable

Symbol	$p(x)$	$q(x)$
a	1/2	1/3
b	1/4	1/3
c	1/4	1/3

Calculate $H(p)$, $H(q)$, $D(p\|q)$ and $D(q\|p)$. Verify that in this case $D(p\|q) \neq D(q\|p)$.

Solution:

$$\begin{aligned} H(p) &= \frac{1}{2} \log 2 + \frac{1}{4} \log 4 + \frac{1}{4} \log 4 = 1.5 \text{ bits} \\ H(q) &= 3 \times \frac{1}{3} \log 3 = 1.58496 \text{ bits} \\ D(p\|q) &= \frac{1}{2} \log \frac{3}{2} + \frac{1}{4} \log \frac{3}{4} + \frac{1}{4} \log \frac{3}{4} = \log 3 - 1.5 = 0.08496 \text{ bits} \\ D(q\|p) &= \frac{1}{3} \log \frac{2}{3} + \frac{1}{3} \log \frac{4}{3} + \frac{1}{3} \log \frac{4}{3} = -\log 3 + \frac{5}{3} = 0.0817 \text{ bits} \end{aligned}$$

It is clear that $D(p\|q) \neq D(q\|p)$.

6. **Discrete entropies**. Let X and Y be two independent integer-valued random variables. Let X be uniformly distributed over $\{1, 2, \ldots, 8\}$, and let $Pr\{Y = k\} = 2^{-k}$, for $k = 1, 2, 3, \ldots$

(a) Find $H(X)$.

(b) Find $H(Y)$.

(c) Find $H(X+Y, X-Y)$.

hint: For (b), the following expression may be useful

$$\sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2} \ .$$

For (c), if you do not find a direct way, try to use

$$\begin{aligned} I(X,Y;X+Y,X-Y) & \\ &= H(X,Y) - H(X,Y|X+Y,X-Y) \\ &= H(X+Y,X-Y) - H(X+Y,X-Y|X,Y). \end{aligned}$$

Solution:

(a) $H(X) = \log 8 = 3$ bits.

(b) $H(Y) = \sum_k 2^{-k} \log 2^k = \sum_k k 2^{-k} = \frac{1/2}{(1-1/2)^2} = 2$ bits.

(c) Since $(X,Y) \rightarrow (X+Y, X-Y)$ is a one-to-one mapping,

$$H(X+Y, X-Y) = H(X,Y) = H(X) + H(Y) = 5 \text{ bits.}$$

Alternatively, use the hint. It is clear $H(X,Y|X+Y,X-Y) = 0$ and $H(X+Y,X-Y|X,Y) = 0$. Hence $H(X+Y,X-Y) = H(X,Y) = H(X) + H(Y) = 5$ bits.