Constraint optimization

A general formula for these problems is

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t.} \quad c_i(x) = 0, \quad i \in \mathcal{E}$$

$$c_i(x) \geq 0, \quad i \in \mathcal{I}$$

where f and the functions c_i are all smooth, real valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indicies.

f - objective function

c_i $(i \in \mathcal{E})$ - equality constraints

c_i $(i \in \mathcal{I})$ - inequality constraints.

We define the feasible set of points x that satisify the constraints

$$\Omega = \{x \mid c_i(x) = 0, i \in \mathcal{E};\ c_i(x) \geq 0, i \in \mathcal{I}\}$$

Simplified as

$$\min_{x \in \Omega} f(x)$$

- A vector x^* is a <u>local solution</u> if $x^* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of x^* such that $f(x) \geq f(x^*)$ for $x \in \mathcal{N} \cap \Omega$

$$\mathcal{N}(x^*) \text{ s.t. } f(x) \geq f(x^*) \ \forall x \in \mathcal{N} \cap \Omega$$

- A vector x^* is a <u>strict local solution</u> (aka strong local solution) if $x^* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of x^* such that $f(x) > f(x^*)$ for all $x \in \mathcal{N} \cap \Omega$ with $x \neq x^*$

$$\mathcal{N}(x^*) \mid f(x) > f(x^*), \forall x \in \mathcal{N} \cap \Omega \mid x \neq x^*$$

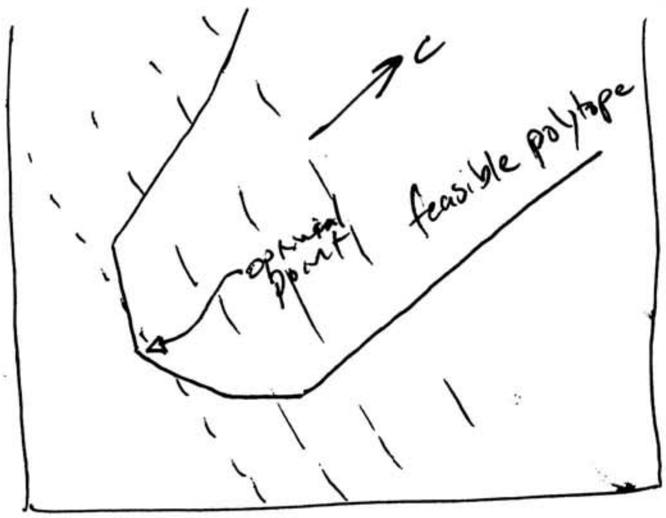
- A point x^* is an <u>isolated local solution</u> if $x^* \in \Omega$ and there is a $\mathcal{N}$ of x^* such that x^* is the only local solution in $\mathcal{N} \cap \Omega$.

$$\mathcal{N}(x^*) \mid x^* \in \mathcal{N} \cap \Omega$$

Linear Programming

- Linear objective function and linear constraints, which may include both equalities and inequalities.

- The feasible set is a polytope (a convex, connected set with flat polygonal faces)



- Contours of objective function are planar

depicts a linear program in 2-D space, in which the contours of the objective function are indicated by dotted lines.

- The problem has no solution, if the feasible set is empty (infeasible case)

- The problem has "   ", if the function is unbounded below on the feasible region (unbounded case)

- Linear programs are usually stated and analyzed in the following <u>standard form</u>:

$$\min_{x} \; c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

where c and x are vectors in $\mathbb{R}^n$, b is a vector in $\mathbb{R}^m$, and A is an $m \times n$ matrix with Rank m (ie. $m < n$, m lin independent)

- Simple ~~device~~ can be used to transform any linear program to <u>standard form</u>:

Ex.

$$\min_{x} \; c^T x$$
$$\text{s.t. } Ax \leq b$$

(i.e. no bounds on x)

Now we can convert the inequality constraint to equalities by introducing a vector of <u>slack varibles</u> z to make

$$\min_{x} \; c^T x$$
$$\text{s.t. } Ax + z = b, \; z \geq 0$$

whats wrong with this form?

Not all variables are constrained to be nonnegative.

We can solve this by splitting x into its nonegative and nonpositive parts,

$$x = x^+ - x^-$$

where $x^+ = \max(x, 0) \geq 0$

$x^- = \max(-x, 0) \geq 0$

Now the problem becomes:

$$\min_{[x^+; x^-; z]^T} \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix}$$

$$\text{s.t.} \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} = b$$

$$\begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} \geq 0$$

Standard Form

• Inequality constraints of the form $x \le u$ or $Ax \ge b$ always can be converted to equality constraints by adding or subtracting slack variables.

$$x \le u \iff x + w = u, \quad w \ge 0$$

$$Ax \ge b \iff Ax - y = b, \quad y \ge 0$$

Add (Slack), subtract (surplus)

$\boxed{(x, z) \text{ primal variables}}$

We can also convert a max $c^T x$ to a min $(-c)^T x$.

Standard Form (Primal),

$$\begin{aligned} &\max \ b^T \lambda \\ &\text{s.t. } A^T \lambda \le c \end{aligned} \quad \text{(dual)}$$

$$\Rightarrow \begin{aligned} &\max \ b^T \lambda \\ &\text{s.t. } A^T \lambda + s = c \\ &\quad s \ge 0 \end{aligned}$$

(λ, s) dual variables.

Strong Duality Theorem

i) If either problem has a (finite) solution, then so does the other, and the objective values are equal.

ii) if either problem is unbounded, then the other problem is infeasible.

(Fundamental Theorem of Linear Programming)

- If primal has a nonempty feasible region, then there is at least one basic feasible point.
- If primal has solution, then at least one such solution is a basic optimal point.
- If primal ~~has~~ is feasible and bounded, then it has an optimal solution

5

All basic feasible points for primal are verticies of the feasible polytope $\{x \mid Ax = b, x \geq 0\}$ and vice versa.

Simplex Method

- The simplex method's strategy of examining only basic feasible points will converge to a solution of the primal only if

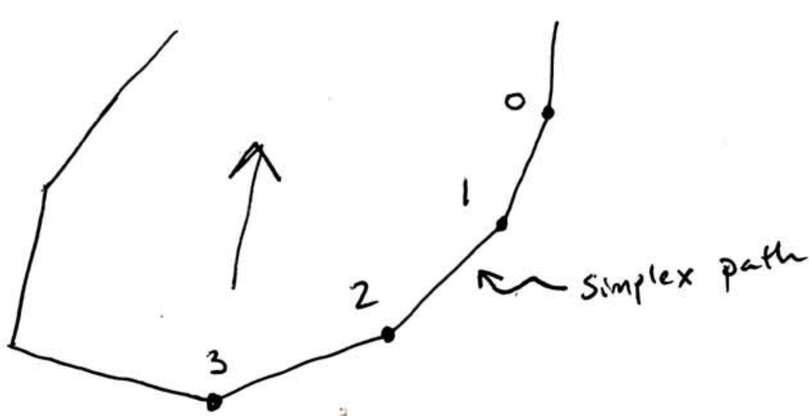
  a) the problem has basic feasible points

  b) at least one such point is a basic optimal point, that is, a solution of the primal that is also a basic feasible point.

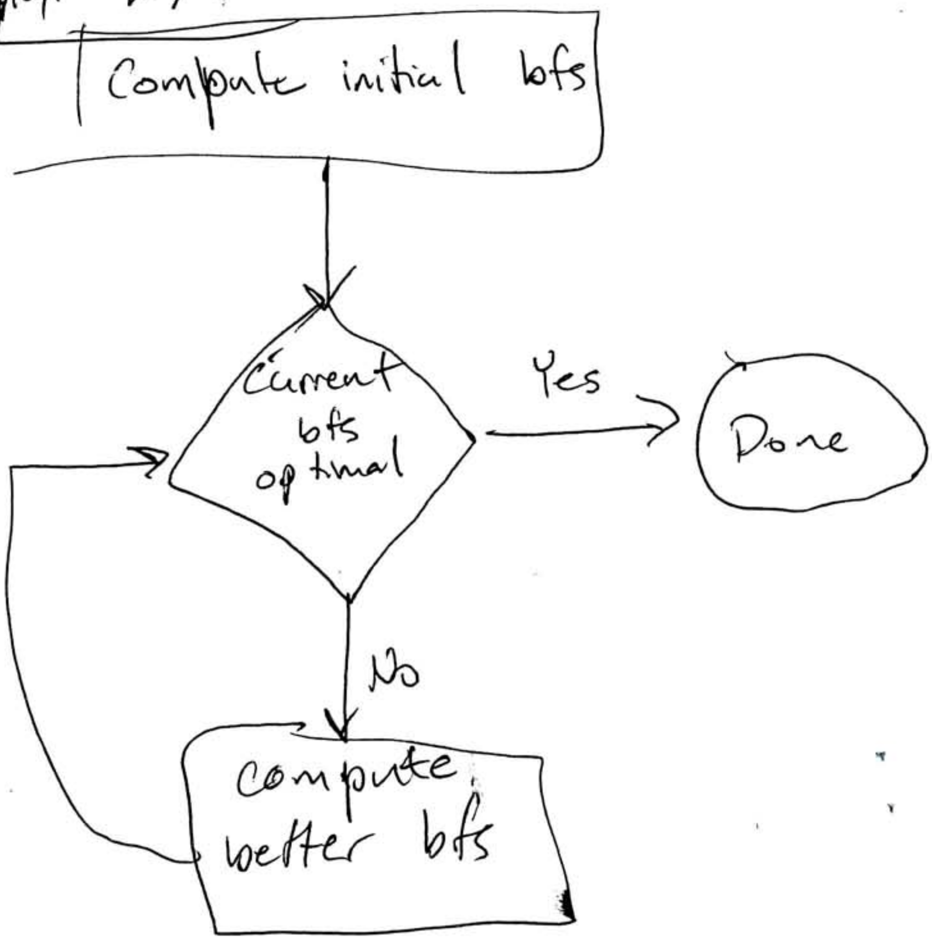
- There are a number of variants of the simplex method

Generalization:

Proceed from one basic feasible solution to another, in such a way as to continually decrease the value of the objective function until a minimum is reached.



"Simplex" Logic



Example :

$$\max \quad Z = X_1 + 3X_2$$

$$\text{s.t.} \quad X_1 + X_2 \le 10$$

$$5x_1 + 2x_2 \le 20$$

$$x_1 + 2x_2 \le 36$$

$$X_1, X_2 \ge 0$$

Slack Variables : (x_3, x_4, x_5)

$$x_1 + x_2 + x_3 = 10$$

$$5x_1 + 2x_2 + x_4 = 20$$

$$x_1 + 2x_2 + x_5 = 36$$

$$Z = x_1 + 3x_2 \longrightarrow -x_1 - 3x_2 + Z = 0$$

initial simplex tableau

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 5 & 2 & 0 & 1 & 0 & 0 & 20 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

pivot column since -3 is most negative indicator

Quotients: $10/1 = 10 \quad 20/2 = 10 \quad 36/2 = 18$

Since we have two equally small quotients, we can choose either entry in column 2 to be the pivot.

Lets choose the number 1 in column 2 to be the pivot.

$$r_2 \to -2r_1 + r_2 \qquad \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$r_3 \to -2r_1 + r_3 \qquad \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 & 16 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$r_4 \to 3r_1 + r_4 \qquad \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 & 16 \\ \hline 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{array}\right]$$

Since there are no negatives in the last row, we are done pivoting.

$$2x_1 + 3x_3 + Z = 30$$

$$\Rightarrow Z = 30 - 2x_1 - 3x_3$$

Basic variables are x_2, x_4, x_5, and Z

Non basic variables are x_1, x_3

Solution is found by setting the nonbasic variables equal to zero.

$$x_1, x_3 = 0$$

then from final tableau we have

$$x_1 + x_2 + x_3 = 10$$

$$3x_1 - 2x_3 + x_4 = 0$$

$$-x_1 - 2x_3 + x_5 = 16$$

$$\rightarrow \quad x_2 = 10, \; x_3 = 0, \; x_5 = 16$$

and max is $Z = 30$ (right lower corner of tableau)

Simplex Method

1. Convert each constraint to an equation by adding slack variables

2. Set up the initial tableau

3. Locate the most negative indicator. This indicator determines the pivot column.

4. Use the positive entries in the pivot column to form the necessary quotients for determining pivot. If there are no positive entries, no maximum solution exists. If two quotients are equally the smallest, let either determine the pivot.

5. Divide the pivot row by the pivot to change the pivot to 1. Then use row operations to change to change all other entries in the pivot column to 0.

6. If the indicators are all positive or zero, you have found the final tableau.

7. Set each nonbasic variable to 0 and solve the
5 system for the basic variables. The max is in lower right

Quadratic Programming

The general QP can be stated as

$$\min_{x} \; q(x) = \tfrac{1}{2} x^T G x + x^T c$$

$$\text{s.t.} \quad Ax = b \qquad i \in \mathcal{E} \quad (\text{Equality constrained})$$

$$\qquad Ax \geq b \qquad i \in \mathcal{I} \quad (\text{Inequality constrained})$$

where G is symmetric $n \times n$

$\mathcal{E}; \mathcal{I}$ finite sets

$(c, x, A), \; i \in \mathcal{E} \cup \mathcal{I}$

Quadratic programs can always be solved (or shown infeasible) in a finite amount of computation. Dependent on problem, strongly.

Convex QP $\rightarrow$ G is pos. semi def. $x^T M x \geq 0$

This case is often ~~strict~~ similarly in difficulty to LP.

Strictly convex QP $\rightarrow$ G is pos. def. $x^T M x > 0$

non convex QP $\rightarrow$ G is indefinite

Solution Methods include

interior point
active set
augmented lagrangian
conjugate gradient
extensions of the simplex method