HW5 Q2

2025

(a) $\overline{Z_5}$ = algebraic closure of Z_5.

$Z_p = \mathbb{F}_p$ = field of p-elements.

$\mathbb{F}_{p^n} \subset \mathbb{F}_{p^m}$ if n divides m $(n|m)$

So, $\mathbb{F}_{5^1} \subseteq \mathbb{F}_{5^k}$ $\forall k > 1$

So the algebraic closure of $\mathbb{F}_5$ = union of all fields $\mathbb{F}_{5^k}$, $k > 1$, that contain $\mathbb{F}_5$.

(b). Let $J = \langle f_1, f_2 \rangle \subseteq Z_5[x, y]$.

According to the weak Null~~st~~tellensatz, if

G = Grobner basis $GB(f_1, f_2) == 1 \Rightarrow$ no solutions, over the closure.

otherwise there are solutions.

So, I used lex term order with $x > y$ & computed GB. $G = GB(f_1, f_2)$. I got:

$$G = \{g_1, g_2, g_3\}, \quad g_1 = y^5 + 2y^4 - y^2 - y + 2$$
$$g_2 = xy - 2x + 2y^3 + 2y$$
$$g_3 = x^2 + y^2 + 1$$

Since $G \neq \{1\}$, it means there are solutions somewhere in the closure $\overline{Z_5}$, but we don't know where.

(c) Since there exist a pure power of both x & y as leading monomials in the GB:

$$lm(g_1) = y^5 \qquad lm(g_3) = x^2$$

$\Rightarrow$ Finite solutions.

So, how many solutions? $\Rightarrow$ # of standard monomials.

$$SM(G) = \{1, x, y, y^2, y^3, y^4\}$$

$$\Rightarrow |SM(G)| = 6 = 6 \text{ solutions}$$

(d) To find solutions over $\mathbb{Z}_5 \equiv \mathbb{F}_5$ itself, restrict the variety to $\mathbb{Z}_5$ by adding vanishing polynomials of the field of $\mathbb{Z}_5$: $\langle x^5 - x, y^5 - y \rangle = J_0$.

So, compute $GB(J + J_0) = G_1$ with lex $x > y$:

$$G_1 = \{y^2 + 1, x + y^2 + 1\}$$

Since $G_1 \neq \{1\}$, again there are solutions.

and they are finite.

$SM = \{1, y\} \Rightarrow$ 2 Solutions in $\mathbb{Z}_5$.