Proof of regular Nullstellensatz.

Given. $J = \langle f_1 \dots f_s \rangle$, f vanishes on $V(J)$.

$\Rightarrow f \in I[V(J)]$

Prove that $\exists\, m \geq 1$, s.t. $f^m \in J$.

<u>Proof:</u> $f^m \in J$ or. $f^m \in \langle f_1, \dots f_s \rangle$

or $f^m = A_1 . f_1 + A_2 f_2 + \dots + A_s f_s$.

<u>Trick:</u> Consider ideal $I := \langle f_1, \dots f_s, 1 - y \cdot f \rangle$

(y = new variable). $I \subseteq F[x_1 \dots x_n, y]$

I claim $\boxed{V(I) = \phi}$. $\Leftarrow$ If true.

$1 \in I$ (week nullstellensatz)

$1 = h_1 f_1 + h_2 f_2 + \dots + h_s f_s + g \cdot (1 - y f)$

$h_i(x_1, \dots x_n, y)$. set $y = 1/f(x_1 \dots x_n)$

$$\Rightarrow 1 = \sum_{i=1}^{s} h_i(x_1, \dots x_n, 1/f) \cdot f_i$$

$$\Rightarrow f^m = \Sigma h_i f_i \quad \Rightarrow f^m \in \langle f_1 \dots f_s \rangle$$

$\boxed{\text{So } f^m \in J}$

Now show that $V(I) = \phi$ $\quad I \subseteq k[x_1, \dots x_n, y]$

where $I = \langle f_1, \dots, f_s, 1 - yf \rangle$

Take a point $(a) = (a_1, \dots a_n, a_{n+1})$

① either $(a_1 \dots a_n)$ = common zero of $f_1 \dots f_s$
$- \in V(J)$

② OR $(\quad)$ = not a common zero.

In ① f vanishes on $V(J)$.

$f(a_1 \dots a_n) = 0$ $\quad$ So $1 - y \cdot f = \underline{\underline{1}}$

So $(a_1 \dots a_n, a_{n+1})$ is not in $V(I)$.

In ② $(a_1 \dots a_n)$ is already NOT in $V(I)$

So $[a_1, \dots a_n, a_{n+1}]$ is not in $V(I)$.