ECE/CS 5740/6740
CAD of Digital Circuits: Logic Synthesis & Optimization

Operations on Boolean functions

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Shannon's Expansion

- $f = xf_x + xf_{x'}$
- When f is +ve unate in x, then, $f = xf_x + f_{x'}$
- When f is -ve unate in x, then, $f = f_x + x'f_{x'}$
- Significance on Shannon's expansion
  - Break a function down to it's co-factors, and operate on co-factors
  - $f \odot g = x(f_x \odot g_x) + x'(f_{x'} \odot g_{x'}), \quad \odot =$ any Boolean operation

Binate Functions

- What if the function is not unate in any variable?
- Then the function is called “binate”
- $f = a \oplus b \oplus c$
- $f_a =$
- $f_{a'} =$
- Does $f_a \supseteq f_{a'}$? Or $f_a \subseteq f_{a'}$?
- What about unateness in variables b, c?

Binate Functions → Unate Cofactors

- Processing of Binate functions may be harder
- Most functions in general are binate
- But their cofactors tend to be unate: successive cofactoring leads to unate sub-functions, so Shannon's expansion helps
  - Often called the "Unate Recursive Paradigm"
  - See example below….

Binate Functions → Unate Cofactors

- $f = a'b'c + a'bc' + abc$ and $g = abc + a'b'c'$
- Are f and g unate in any variable?
- $f \cdot g = a f_a g_a + a' f_{a'} g_{a'}$
- $f_a =$ $\qquad\qquad$ $f_{a'} =$
- $g_a =$ $\qquad\qquad$ $g_{a'} =$
- Is f_a +ve unate in b? $(f_{ab} \supseteq f_{ab'})$?
- Is g_a +ve unate in b? $(g_{ab} \supseteq g_{ab'})$

Boolean Function Operations: Boolean Difference

- Boolean difference or Boolean derivative:
  - Is f sensitive to changes in x?
- Denoted: $\dfrac{\partial f}{\partial x} = f_x \oplus f_{x'}$
- If $f_x \oplus f_{x'} = 0$, then $f_x = f_{x'}$
- When is $f_x = f_{x'}$?
- $f = ab + a'b$
- $f = ab + ac + bc$, $f_a \oplus f_{a'} = b'c + bc'$
- As a changes, f changes if $b'c + bc' = \text{TRUE}$

Depict on the circuit — in the classroom, on the board

Consensus of f w.r.t. x

- Consensus of a function f w.r.t. x: $C_x(f) = f_x \cdot f_{x'}$
- Product of cofactors
- Also called universal quantification of f w.r.t. x $= \forall_x\ (f)$
- Gives the largest function, smaller than f, contained in f, which does not have x in it's support
  - Represents the component in f independent of x
  - $f = ab + bc + ac$, consensus w.r.t. $a =$

Depict on a 3D-cube ....

Smoothing of f w.r.t. x

- Smoothing of a function f w.r.t. x: $S_x(f) = f_x + f_{x'}$
- OR of cofactors
- Also called existential quantification of f w.r.t. x $= \exists_x (f)$
- Gives the smallest function larger than f, that contains f, but does not have x in its support
- Makes the function independent of the variable
- $f = ab + ac + bc, \quad \exists_a f = ?$

Depict on a 3D-cube ….

How to Perform Containment Check?

- Logic synthesis, test and verification often requires a check for containment
  - If p, q are Boolean functions, then is $p \subseteq q$?
  - Containment is an implication: $(p \subseteq q) \;\equiv\; (p \implies q)$
  - Example: Unateness, does $f_x \supseteq f_{x'}$?
- A function is TAUTOLOGY if it is TRUE everywhere ($f = \mathbf{1}$)
- Containment check: $(p \implies q)$ if and only if $(p' + q) = \mathbf{1}$
- Solve: $p = bc, q = b + c : \; p \subseteq q$?

Generalized Cofactors and Orthonormal Expansion

- Let $f,\ \phi_i\ (i = 1,\ldots,k)$ be Boolean functions such that
  - $\sum_{i=1}^{k} \phi_i = 1$, and $\phi_i \cdot \phi_j = 0,\ \forall i \neq j$
  - Then we have:

    $$f = \sum_{i=1}^{k} \phi_i \cdot f_{\phi_i} = \phi_1 \cdot f_{\phi_1} + \phi_2 \cdot f_{\phi_2} + \cdots + \phi_k \cdot f_{\phi_k}$$

- Functions f_{ϕ_i} = generalized cofactor of f w.r.t. ϕ_i
- Special case: Let f, g be arbitrary Boolean functions. Then:

$$f = g \cdot f_g + \overline{g} \cdot f_{\overline{g}}$$

Generalized Cofactor

- What is a generalized cofactor and how do we compute one? Given f, g how to compute f_g?

- Generalized cofactors are not unique but they satisfy the following bounds:

- $$f \cdot g \subseteq f_g \subseteq f + g'$$

- Many functions may satisfy the above bound and any one of them may be taken as f_g

- Another view:

- $$\underbrace{f \cdot g}_{\text{ON-set}} \subseteq f_g \subseteq \underbrace{f + g'}_{\text{ON-set} \cup \text{DC-set}}$$

- Solve examples on the board, using K-maps, and see lecture notes