EX: Find a radially symmetric joint probability density function, $f(x, y)$, for which X and Y are independent. That is, find an $f(x, y)$ that may be written as a function of $x^2 + y^2$. Hint: consider what type of function turns multiplication into addition.

SOL'N: The function that turns multiplication into addition is the exponential:

$$e^x e^y = e^{x+y}$$

We replace x and y with x^2 and y^2 to obtain a function of $x^2 + y^2$. We then add a minus sign to obtain functions that have finite area over the interval $(-\infty, \infty)$. Finally, we need a normalizing constant to make the area of each function equal to one so we have valid probability density functions.

Calculating the normalizing constant by calculating the integral of e^{-x^2} directly requires advanced complex analysis. Instead, we observe that we have a gaussian density function with $\sigma^2 = \frac{1}{2}$:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

Thus, we have the following $f_X(x)$ and $f_Y(y)$:

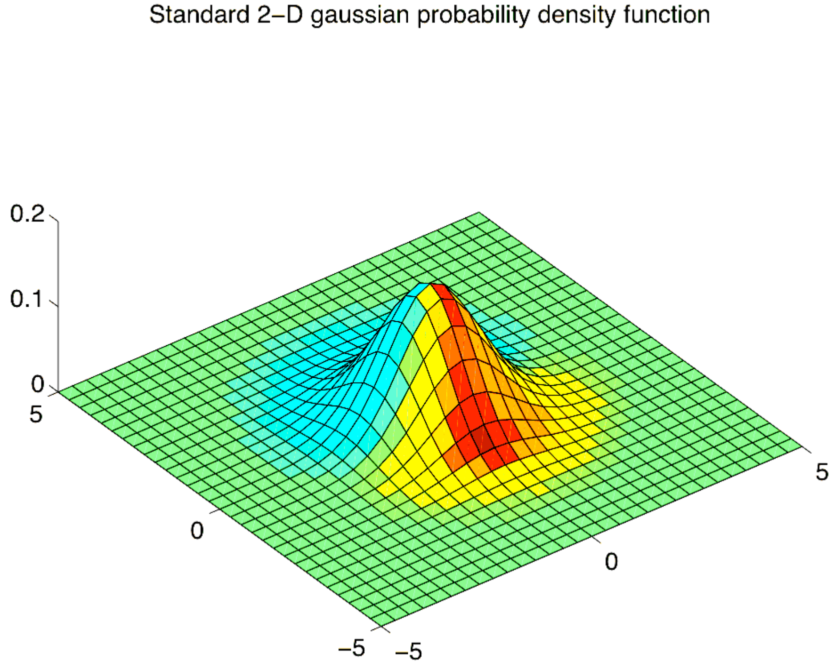
$$f_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \qquad f_Y(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}$$

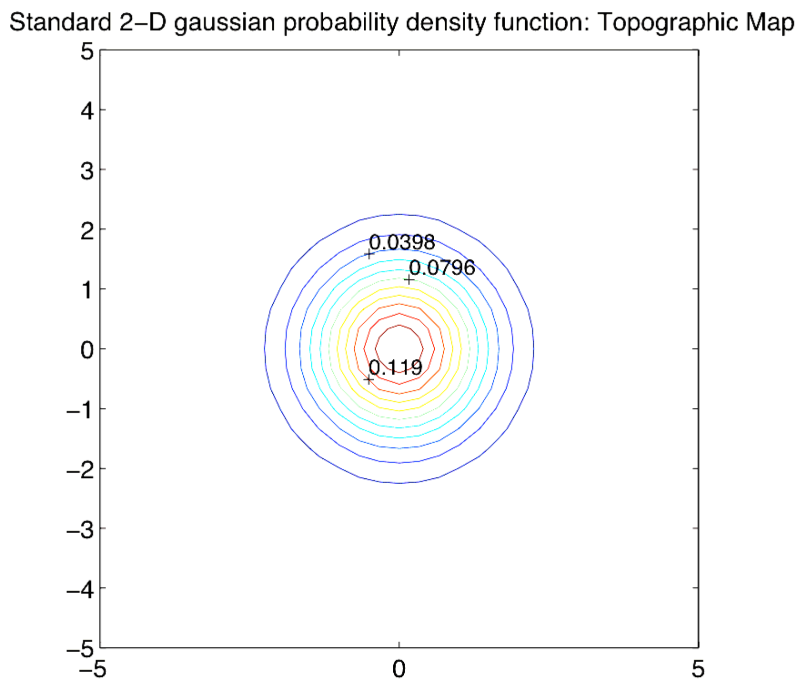
Since X and Y are independent, the probability density function $f(x, y)$ is the product of $f_X(x)$ and $f_Y(y)$:

$$f(x,y) = \frac{1}{\pi} e^{-(x^2+y^2)}$$

This definition holds for all real x and y. The plots below show the shape of this 2-dimensional gaussian (but with $\sigma^2 = 1$). Note the circular symmetry in the contour plot.

What is remarkable about the circularly symmetric gaussian is that, since x and Y are independent, every cross section must have the same shape after being scaled vertically to achieve an area equal to one. A cylindrically-shaped $f(x, y)$ would have cross sections of different widths, for example.


Standard 2−D gaussian probability density function
0.2
0.1
0
5
0
−5
−5
0
5


Standard 2−D gaussian probability density function: Topographic Map
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
0
5
0.0398
0.0796
0.119