EX: If $f(x, y) = f_X(x)f_Y(y)$, show by symbolic calculation of the appropriate integral whether the following statement is true:

$$\mathrm{P}(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = \mathrm{P}(x_1 \le X \le x_2)\mathrm{P}(y_1 \le Y \le y_2)$$

SOL'N: We start with the left side of the equation and show that we can transform it into the right side of the equation. Our first step is to write the probability on the left side in terms of the joint probability density function, $f(x, y)$:

$$P(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)dxdy$$

We substitute for $f(x, y)$ to obtain an expression that we can separate into an integral over x and an integral over y:

$$P(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_X(x)f_Y(y)dxdy$$

or

$$P(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = \int_{y_1}^{y_2} f_Y(y)\int_{x_1}^{x_2} f_X(x)dxdy$$

or

$$P(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = \int_{y_1}^{y_2} f_Y(y)dy\int_{x_1}^{x_2} f_X(x)dx$$

Reversing the order of multiplication gives

$$P(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = \int_{x_1}^{x_2} f_X(x)dx\int_{y_1}^{y_2} f_Y(y)dy.$$

Now we use the following identities:

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f_X(x)dx$$

and

$$P(y_1 \le Y \le y_2) = \int_{y_1}^{y_2} f_Y(y)dy$$

Substituting these yields the equation given in the problem, and we are done:

$$P(x_1 \le X \le x_2 \text{ and } y_1 \le Y \le y_2) = P(x_1 \le X \le x_2)P(y_1 \le Y \le y_2).$$