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ECE 3530 Poisson Process

30 Mar 04

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def: Poisson process ≡ 1) Involves measurements of times when events occur.

2) The past timing pattern of events has no influence on timing of future events; no memory; independence of event times.

3) The probability density that an event will occur at a particular time is the same for all times; no clock; uniform likelihood of event times.

4) Events occur at some average rate; the ave # of events in time Δt is proportional to the duration of Δt.

ex: Leaves falling from a tree in autumn when there is no wind. The times when leaves hit the ground approximates a Poisson process.

Note: in many situations we can argue that a process is influenced by variables that cause event times to be dependent on one another, (such as one falling leaf knocking off another one), but if the dependence is weak we may still gain advantage by assuming our process is a Poisson process.

Note: even if conditions for a process change, (a wind blows more leaves off), we may be able to assume a Poisson process for an interval of time, (while the wind is steady).

ECE 3530 Poisson Process (cont.)

30 Mar 04
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def: rate of Poisson process $\equiv \lambda \equiv$ ave # events per unit time

tool: mean number of events per unit time for Poisson process $= \lambda$

tool: $E\{\#\text{ events in time } t\} = \lambda t \equiv \mu$

tool: $P\{n \text{ events in time } t\} = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$

def: " $\equiv$ Poisson distribution

check: $\sum_{n=0}^{\infty} P\{n \text{ events in time } t\} = \sum_{n=0}^{\infty} \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$

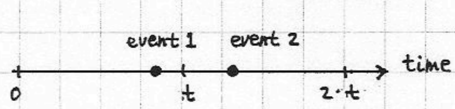
sum must = 1 since total prob will equal 1 $\quad = e^{-\lambda t} \sum_{n=0}^{\infty} \dfrac{(\lambda t)^n}{n!}$

Recall Taylor series for e^x is $\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ $\quad = e^{-\lambda t} e^{\lambda t}$

$= 1$ ✓

check: We independence for the number of events in adjacent intervals. For two intervals of length t side-by-side we have

$$P\{2 \text{ events in time } 2\cdot t\} = \frac{e^{-2\lambda t}(2\lambda t)^2}{2!}$$



Given independence, we have

$$\begin{aligned} P\{2 \text{ events in time } 2\cdot t\} = &(P\{2 \text{ events in time } t\} \\ &\cdot P\{0 \text{ " " " "}\}) \\ &+ (P\{1 \text{ " " " "}\} \\ &\cdot P\{1 \text{ " " " "}\}) \\ &+ (P\{0 \text{ " " " "}\} \\ &\cdot P\{2 \text{ " " " "}\}) \end{aligned}$$

We consider # events in 1^{st} and 2^{nd} intervals (and take product of probabilities for 1^{st} and 2^{nd} intervals) and sum probs of scenarios.

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ECE 3530 Poisson Process (cont.)

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So we have
$$\frac{e^{-\lambda t}(\lambda t)^2}{2!}\cdot\frac{e^{-\lambda t}(\lambda t)^0}{0!} + \frac{e^{-\lambda t}(\lambda t)^1}{1!}\cdot\frac{e^{-\lambda t}(\lambda t)^1}{1!} + \frac{e^{-\lambda t}(\lambda t)^0}{0!}\cdot\frac{e^{-\lambda t}(\lambda t)^2}{2!}$$

$$= e^{-2\lambda t}\left[\frac{(\lambda t)^2}{2}\cdot 1 + \frac{\lambda t}{1}\cdot\frac{\lambda t}{1} + 1\cdot\frac{(\lambda t)^2}{2}\right]$$

$$= e^{-2\lambda t}\, 2(\lambda t)^2$$

$$= \frac{e^{-2\lambda t}(2\lambda t)^2}{2!}$$

$$= P\{2 \text{ events in time } 2\cdot t\} \checkmark$$

tool: Poisson distribution with rate $\lambda = \mu$ $= \lim_{n\to\infty}$ (Binomial distribution with $np = \lambda = \mu$)

justification:

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 0&1&0&1&1&0&0&0&1&0&1&0&1&0&0&1&1&1&0&1\\\hline\end{array}}_{n \text{ time slices}}$$

$t=0$ $t=1$

Divide one unit of time into n slices.
Write a 0 if no event occurs in a given slice.
" " 1 " an event does occur " " " ".

We expect to have $\# 1\text{'s} = \lambda$
So $P\{1 \text{ in a slice}\} = \frac{\lambda}{n} \equiv p$ or $np = \lambda$

$P\{\lambda \text{ events in } n \text{ slices}\} = \binom{n}{\lambda} p^{\lambda}(1-p)^{n-\lambda}$
= Binomial distribution

The difference between Poisson and the Binomial is that there could be 2 events in a slice for Poisson. If we take $n\to\infty$, the probability of this approaches zero, and the Binomial approach the Poisson dist.

ECE 3530 Poisson Process (cont.)

30 Mar 04
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tool: $\lim_{\lambda \to 0}$ (Poisson distribution for $n=1$ event in time $t=1$)

= Binomial distribution for success in one trial for $p = \lambda$

$$= \binom{1}{1} p^1 \cdot (1-p)^0 = p = \lambda$$

pf: Poisson distribution for $n=1$ event in time $t=1$

$$= \left. \frac{e^{-\lambda t} (\lambda t)^1}{1!} \right|_{t=1} = e^{-\lambda} \lambda$$

Now $\lim_{\lambda \to 0} e^{-\lambda} \cdot \lambda = \lim_{\lambda \to 0} \left(\frac{1}{0!} - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \ldots \right) \lambda$

$= \lim_{\lambda \to 0} \frac{\lambda}{0!} - \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} - \ldots$

$= \lim_{\lambda \to 0} \lambda$ acts like λ as $\lambda \to 0$ because higher order terms small

Note: A more precise statement is that the Poisson distribution behaves asymptotically like λ as $\lambda \to 0$, (for $n=1$ event)

tool: *probability density of intervals between events for Poisson process* $\equiv p(\Delta t) = \lambda e^{-\lambda \Delta t}$ $\equiv$ *exp dist* where λ = rate of process $\equiv$ ave events per unit time.

pf: $p(\Delta t) = \frac{d}{d\Delta t} P\{\text{interval} \le \Delta t\} = \frac{d}{d\Delta t}\left(1 - P\{0 \text{ events in } \Delta t\}\right)$

$$\text{"} = \frac{d}{d\Delta t} \frac{\left(1 - e^{-\lambda \Delta t}\right)(\lambda \Delta t)^0}{0!} = \frac{d}{d\Delta t}\left(1 - e^{-\lambda \Delta t}\right)$$

$$p(\Delta t) = \lambda e^{-\lambda \Delta t}.$$

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ECE 3530 Poisson Process (cont.)

30 Mar 04
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tool: probability density of interval Δt_n required for n events to occur in Poisson process with rate λ

$$\equiv p(\Delta t_n) = \frac{\lambda^n}{(n-1)!} (\Delta t_n)^{n-1} e^{-\lambda \Delta t_n} \equiv \text{gamma dist}$$

$\alpha = n,\ \beta = \frac{1}{\lambda}$

pf: (for $n=2$)

$$p(\Delta t_2) = \frac{d}{d\Delta t_2} P\{\text{interval for 2 events} \le \Delta t_2\}$$

$$= \frac{d}{d\Delta t_2} 1 - (P\{0 \text{ events in } \Delta t_2\} + P\{1 \text{ event in } \Delta t_2\})$$

$$= \frac{d}{d\Delta t_2} 1 - \left(e^{-\lambda \Delta t_2} \frac{(\lambda \Delta t_2)^0}{0!} + e^{-\lambda \Delta t} \frac{(\lambda \Delta t_2)^1}{1!}\right)$$

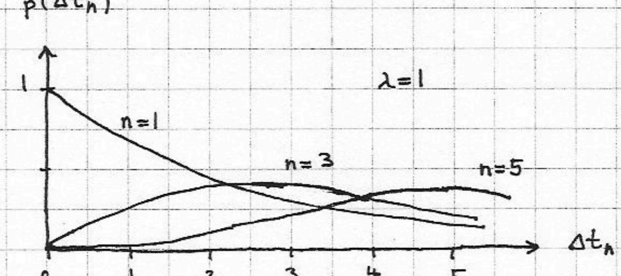
$$= \frac{d}{d\Delta t_2} 1 - e^{-\lambda \Delta t_2}(1 + \lambda \Delta t_2)$$

$$= -\left[-\lambda e^{-\lambda \Delta t_2}(1 + \lambda \Delta t_2) + e^{-\lambda \Delta t_2} \lambda\right]$$

$$= e^{-\lambda \Delta t_2}\left[\lambda(1 + \lambda \Delta t_2) - \lambda\right]$$

$$= e^{-\lambda \Delta t_2} \lambda^2 \Delta t_2$$

$$= \frac{\lambda^2}{(2-1)!} (\Delta t_2)^{2-1} e^{-\lambda \Delta t} \checkmark$$



30 Mar 04
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31 Mar 04
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tool: $E\{\Delta t_n\} = \dfrac{n}{\lambda}$ for Poisson process of rate λ

Note: The proof is by direct calculation, (tedious but straightforward).

tool: $\sigma^2_{\Delta t_n} = \dfrac{n}{\lambda^2}$ for Poisson process of rate λ

Note: Again, found by direct calculation.

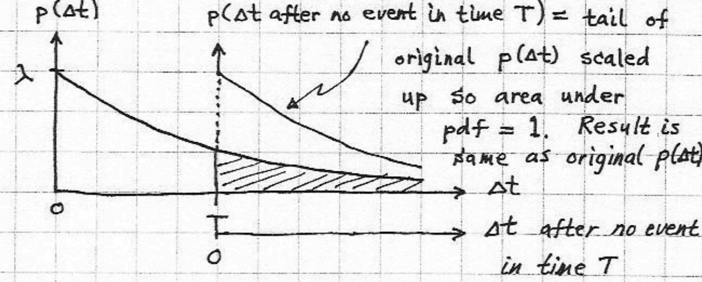
tool: $$\int x^n e^{ax}\,dx = e^{ax}\left[\frac{x^n}{a} - \frac{n x^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} - \ldots + (-1)^{n-1}\frac{n!\,x}{a^n} + (-1)^n \frac{n!}{a^{n+1}}\right]$$

where $n \ge 0$

ref: Tables of Integrals and other Mathematical Data Herbert Bristol Dwight, MacMillan: NY, 1961.

tool: If no event occurs for time T in a Poisson process, the density function for the interval, Δt, to the next event is the same as the orignal $p(\Delta t)$, namely $p(\Delta t) = \lambda e^{-\lambda \Delta t}$. Thus, the process is <u>memoryless</u>. We may also use this observation to derive $p(\Delta t)$ to begin with.

pict:



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ECE 3530 Poisson Process (cont.)

30 Mar 04

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ex: An astronomer and a climatologist have determined that an asteroid large enough to cause mass extinctions hits the earth on average once every 10 million years.

Assuming the arrivals of these asteroids follow a Poisson process, answer the following questions:

a) If $X \equiv$ interval between large asteroids, find $E\{X\}$ and σ_X^2

b) Find $P\{X \geq 10$ million years$\}$. X as in (a) Is it $1/2$?

c) Find $P\{2$ large asteroids hit earth in next 1 million years$\}$.

d) Find the probability density function for the interval Δt_3 for 3 large asteroids to hit earth. Locate the peak value as well.

soln: a) $E\{X\} = \dfrac{n=1}{\lambda = 1/10^7 \text{ yrs}} = 10^7 \text{ yrs} \equiv \mu_X$

$$\sigma_X^2 = \frac{n=1}{\lambda^2} = 10^{14} \text{ yrs}^2$$

So $\sigma_X = 10^7$ yrs $= \mu_X$. (Always, if $n=1$)

b) $p_X(x) = \lambda e^{-\lambda x}$ exponential distribution when $n=1$. Here $\lambda = \frac{1}{10^7}$

$$P\{X \geq 10 \text{ million yrs}\} = \int_{10^7}^{\infty} p_X(x)\,dx$$

$$= \int_{10^7 = \frac{1}{\lambda}}^{\infty} \lambda e^{-\lambda x}\,dx = \lambda \left.\frac{e^{-\lambda x}}{-\lambda}\right|_{\frac{1}{\lambda}}^{\infty}$$

$$= e^{-1}$$

$$P\{X \geq 10 \text{ million yrs}\} = \frac{1}{e} \approx \frac{1}{3}$$

So, more often than not, the interval between asteroids is less than 10^7 yrs, yet the average interval is 10^7 yrs.

Why? There are a few very long intervals that increase the average.

c) $P\{2 \text{ large asteroids in } 1 \text{ million yrs}\}$

$$= \int_0^{10^6} p(\Delta t_2)\,d\Delta t_2$$

$$= \int_0^{10^6} \frac{\lambda^2 (\Delta t_2)^{2-1}}{(2-1)!} e^{-\lambda \Delta t_2}\,d\Delta t_2$$

$$= \int_0^{10^6} \frac{\lambda^2 x}{1!} e^{-\lambda x}\,dx \qquad \text{use } x \text{ for } \Delta t_2$$

$= \ldots$ Wait, This is the hard way!

Use Poisson distribution instead:

$$P\{2 \text{ events in } t = 10^6\} = \frac{e^{-\lambda t}(\lambda t)^2}{2!}$$

where $\lambda = \frac{1}{10^7}$ and $\lambda t = \frac{1}{10}$

$$P\{2 \text{ large asteroids in } 10^6 \text{ yrs}\} = \frac{e^{-1/10}}{200} \approx \frac{1}{200}$$

U U

ECE 3530 Poisson Process (cont.)

30 Mar 04
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d) $p(\Delta t_3) = \frac{\lambda^3}{(3-1)!} (\Delta t_3)^{3-1} e^{-\lambda \Delta t_3}$

where $\lambda = \frac{1}{10^7 \text{ yrs}}$

$p(\Delta t_3) = \lambda^3 \Delta t_3^2 e^{-\lambda \Delta t_3}$

$p(\Delta t_3) = \frac{\Delta t_3^2}{10^{21}} e^{-\Delta t_3 / 10^7}$

To find max (or peak) use time scale of 1 time unit $= 10^7$ yrs so $\lambda = 1$.

Then $p(\Delta t_3) = \Delta t_3^2 e^{-\Delta t_3}$

where $\Delta t_3 = \text{time}/10^7$ yrs

max $p(\Delta t_3)$ when $\frac{d}{d\Delta t_3} p(\Delta t_3) = 0$.

Use x for Δt_3.

$\frac{d}{dx} x^2 e^{-x} = 0 = 2x e^{-x} - x^2 e^{-x}$

$\therefore 2 - x = 0 \Rightarrow x = 2$ or $\Delta t_3 = 2 \cdot 10^7$ yrs

So $p(\Delta t_3)$ peaks at 20 million yrs $= 2\mu$.