2 Mar
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Nonlinear Functions of Random Variables

tool: For an strictly increasing function $g(X) = Y$, the probability density function for Y is given by

$$p_{Y=g(X)}(y) = p_X\left(g^{-1}(y)\right) \cdot \frac{d}{dy} g^{-1}(y)$$

where $g^{-1}(y)$ is inverse function for $g(x)$

Note: $g^{-1}(y)$ does not mean $1/g(y)$.

Instead, if $Y = e^X$, for example, then $X = \ln Y$, and $g^{-1}(y) = \ln y$, (not $1/e^y$).

pf: Use the cumulative distribution function, $F_X(x)$:

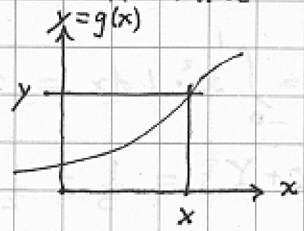
$$F_X(x) \equiv P\{X \le x\} = \int_{-\infty}^{x} p_X(x)\,dx$$

Given $Y = g(X)$ where $g(X)$ is a strictly increasing function we have:

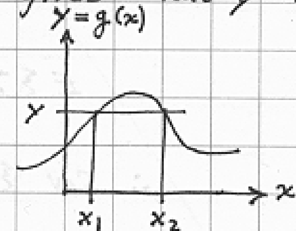
$$F_Y(y) \equiv P\{Y \le y\} = P\{g(X) \le y\} = P\{X \le g^{-1}(y)\}$$

since $X = g^{-1}(g(X))$. Note that we can invert $g()$ since it is strictly increasing. In other words, for every x there is one y value, and for every y value there is one and only one x value that gives that y value.

picture:



unique x for each y, $g(x)$ strictly increasing



not unique x for each y, $g(x)$ not strictly increasing

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pf: (cont.) Now $P\{X \le g^{-1}(y)\} = \int_{-\infty}^{g^{-1}(y)} p_X(x)\,dx = F_X(g^{-1}(y))$.

Since $p_Y(y) = \frac{d}{dy} F_Y(y)$, we have

$$p_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \frac{d}{dy}\int_{-\infty}^{g^{-1}(y)} p_X(x)\,dx$$

We use a change of variables:

$$y = g(x) \qquad x = g^{-1}(y) \qquad \frac{dx}{dy} = \frac{d}{dy} g^{-1}(y)$$

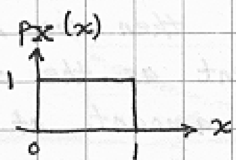
$$y = g(g^{-1}(y)) = y \qquad \text{or } dx = \left[\frac{d}{dy} g^{-1}(y)\right] dy$$

$$\therefore\ p_Y(y) = \frac{d}{dy}\int_{-\infty}^{y} p_X(g^{-1}(y))\,\frac{d\,g^{-1}(y)}{dy}\,dy$$

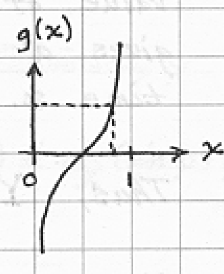
The derivative of the integral is just the integrand.

$$\boxed{\therefore\ p_Y(y) = p_X(g^{-1}(y))\,\frac{d\,g^{-1}(y)}{dy}}$$

ex: $X \sim u(0,1)$
uniform distribution



Consider $y = \tan\left[\pi\left(x-\frac{1}{2}\right)\right] \equiv g(x)$



$g(x)$ maps $(0,1)$ to $(-\infty,\infty)$

Find $g^{-1}(y)$: $\tan^{-1} y = \pi\left(x-\frac{1}{2}\right)$ $\quad \therefore x \equiv g^{-1}(y) = \frac{\tan^{-1} y}{\pi} + \frac{1}{2}$

Since every value of $y = g(x)$ comes from an x in $(0,1)$, it follows that $g^{-1}(y)$ is in $(0,1)$ and $p_X(g^{-1}(y)) = 1$.

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy}\left(\frac{\tan^{-1} y}{\pi} + \frac{1}{2}\right) = \frac{1}{\pi}\,\frac{1}{1+y^2} \qquad \therefore\ p_Y(y) = \frac{1}{\pi}\,\frac{1}{1+y^2}$$

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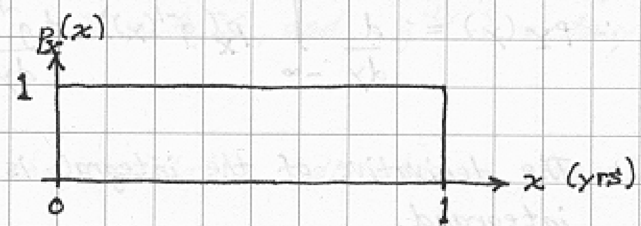
ex: In the course of financial transactions, a bank rounds off to the nearest lower cent if an interest bearing account has interest that is not an exact number of cents. If the bank fails to track these small amounts, once the loss to the customer grows exponentially. How much will the average customer lose?

Assume bank accounts are opened at a uniform rate. If the date when the bank dropped the fractional part of the cent was 1 year ago, we have a uniform probability density function for $X \equiv$ time since customer's fractional cent dropped.



For the sake of simplicity, assume that the rate of interest results in an increase in value of $2e^x$. If the size of the fractional cent is uniformly distributed between 0 and 1, as one would expect, then we may use the expected value of $\frac{1}{2}$ cent as the original amount. This gives a final amount of $y = \frac{2e^x}{2} = e^x$ after time x.

Thus, $Y = e^X \equiv$ amount customer loses for time X.

Now we calculate the average expected loss:

$$E\{Y\} = E\{e^X\} = \int_{-\infty}^{\infty} e^x p_X(x)\,dx = \int_0^1 e^x \cdot 1\,dx$$

$$'' = e^x\Big|_0^1 = e - 1$$

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ex: (cont.) Now the bank is interested in knowing how the losses are distributed among its customers. In other words, what is the probability density function for Y?

We have $Y = g(X) = e^{X}$. $\quad g^{-1}(Y) = \ln Y = X$

$$\therefore\ p_{Y=g(X)}(y) = p_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= p_X(\ln y) \cdot \frac{d}{dy} \ln y$$

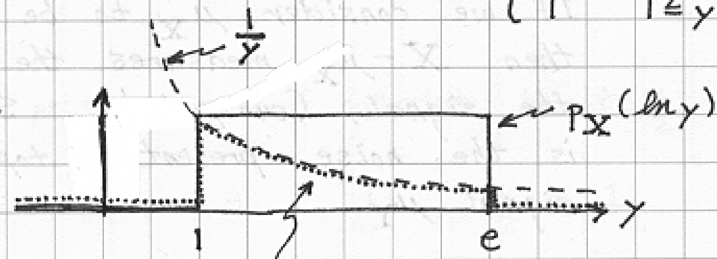
Now $\frac{d}{dy} \ln y = \frac{1}{y}$

and $p_X(\ln y) = \begin{cases} 0 & \ln y < 0 \text{ or } \ln y > 1 \\ 1 & 0 \le \ln y \le 1 \end{cases}$.

uniform distribution on $(0,1)$

Or we can say $p_X(\ln y) = \begin{cases} 0 & y < 1 \text{ or } y > e \\ 1 & 1 \le y \le e \end{cases}$

pict:



$p_Y(y) = p_X(\ln y) \frac{d}{dy} \ln y$

$$\therefore\ p_Y(y) = \begin{cases} 0 & y < 1 \text{ or } y > e \\ \frac{1}{y} & 1 \le y \le e \end{cases}$$

Is the expected value for Y the same for this probability density function as we calculated before?

$$E\{Y\} = \int_{-\infty}^{\infty} y\, p_Y(y)\, dy = \int_1^e y \cdot \frac{1}{y}\, dy = y \Big|_1^e = e - 1$$ Yes