N-dim gaussian

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Given:
Variables $X_1, \ldots, X_N$

Covariance matrix $$K_X \equiv \begin{bmatrix} \sigma_{X_1X_1} & \cdots & \sigma_{X_1X_N} \\ \vdots & & \vdots \\ \sigma_{X_NX_1} & \cdots & \sigma_{X_NX_N} \end{bmatrix}$$

where $\sigma_{X_iX_j} \equiv E\{(X_i-\mu_i)(X_j-\mu_j)\} = E\{X_iX_j\} - \mu_i\mu_j$

$\mu_i \equiv E\{X_i\}$

N-dim gaussian pdf:

$$p(x_1, \ldots, x_N) = \frac{1}{\sqrt{2\pi}^N \sqrt{|K_X|}} e^{-\frac{1}{2}[(x_1,\ldots,x_N)-(\mu_1,\ldots,\mu_N)] K_X^{-1} [(x_1,\ldots,x_N)-(\mu_1,\ldots,\mu_N)]^T}$$

where $|K_X| \equiv$ determinant of K_X

$K_X^{-1} \equiv$ matrix inverse of K_X

N-dim linear transformation of N-dim gaussian pdf:

Consider $$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \quad \text{or} \quad \vec{y} = A\vec{x} + \vec{b}$$

where $A \equiv N \times N$ matrix

$b_i \equiv$ constant

Then $$\begin{bmatrix} \mu_{Y_1} \\ \vdots \\ \mu_{Y_N} \end{bmatrix} = A \begin{bmatrix} \mu_{X_1} \\ \vdots \\ \mu_{X_N} \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \quad \text{or} \quad \vec{\mu}_Y = A\vec{\mu}_X + \vec{b}$$

and $$K_Y \equiv \begin{bmatrix} \sigma_{Y_1Y_1} & \cdots & \sigma_{Y_1Y_N} \\ \vdots & & \vdots \\ \sigma_{Y_NY_1} & \cdots & \sigma_{Y_NY_N} \end{bmatrix} = A K_X A^T$$

$$p(y_1,\dots,y_N) = \frac{1}{\sqrt{2\pi}^N \sqrt{|K_Y|}} e^{-\frac{1}{2}[(y_1,\dots,y_N)-(\mu_{Y_1},\dots,\mu_{Y_N})] K_Y^{-1} [(y_1,\dots,y_N)-(\mu_{Y_1},\dots,\mu_{Y_N})]^T}$$

Transforming N-dim gaussian to independent gaussian R.V.'s:
(as in principal component analysis)

If K_X has distinct nonzero eigenvalues, then

we can diagonalize K_X:

$$K_X = S\Lambda_X S^{-1} = U\Lambda_X U^{T^*} = U\Lambda_X U^T \text{ since } K_X \text{ real, symmetric}$$

where $\Lambda_X \equiv \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \equiv$ eigenvalues of K_X on diagonal

$S \equiv$ columns = eigenvectors of K_X
(col i = eigenvector of λ_i)

$U \equiv$ columns = eigenvectors of K_X normalized to have length = 1
(col i = eigenvector of λ_i)

Note: U is unitary matrix: $U^{T^*} = U^{-1}$

$* \equiv$ complex conjugate ($U^{T^*} \equiv U^H$ common notation)

We also have $\Lambda_X = \sqrt{\Lambda_X}\, I \sqrt{\Lambda_X}^T$

where $\sqrt{\Lambda_X} = \sqrt{\Lambda_X}^T \equiv \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_N} \end{bmatrix}$

Note: K_X real, symmetric $\Rightarrow$ all λ_i real > 0.

It follows that if $\vec{y} = A\vec{x} + \vec{b}$

where $A = \sqrt{\Lambda_X}^{-1} U^T$

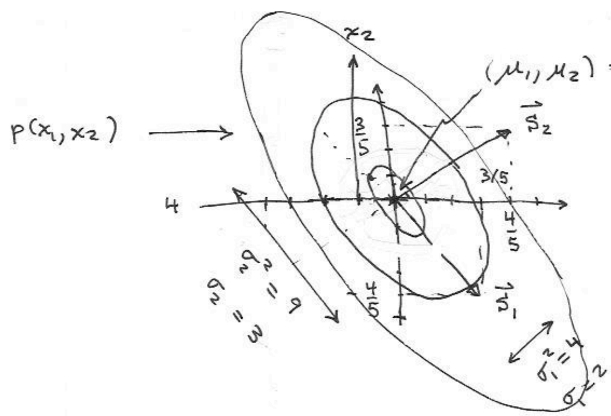
$\vec{b} = -A\vec{\mu}_X$

then $Y_1,\dots,Y_N$ are independent standard gaussian R.V.'s,
(i.e., with $\mu_{Y_i} = 0$ and $\sigma_{Y_i}^2 = 1$).

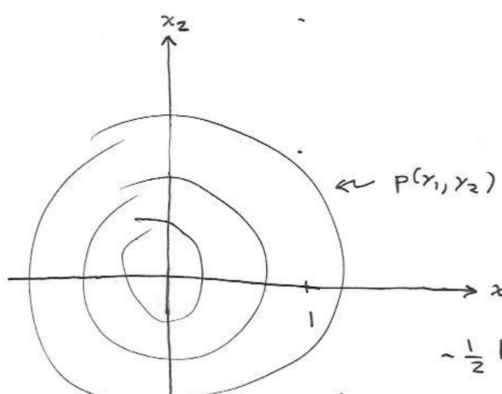
ex: $S = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$ $\Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$ $\vec{\mu}_X = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$S S^T = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S \Lambda S^T = K = \begin{bmatrix} 5.8 & -2.4 \\ -2.4 & 7.2 \end{bmatrix} \qquad K^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{30} \\ \frac{2}{30} & \frac{29}{180} \end{bmatrix}$$



$\vec{s}_i \equiv i^{th}$ eigenvector
" $= i^{th}$ column of S
" = principal axis of $p(\vec{x})$



Whitening transformation
(i.e., creating ind standard gaussian Y_1 and Y_2)

$$\sqrt{\Lambda}^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\vec{y} = A\vec{x} + \vec{b}$$

$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \quad \vec{b} = -A \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$p(x_1, x_2) = \frac{1}{\sqrt{2\pi}^2} \frac{1}{\underset{\underset{36}{\shortparallel}}{\sqrt{|A|}}} e^{-\frac{1}{2} [x_1 - 1, \, x_2 - 4] \begin{bmatrix} \frac{1}{5} & \frac{1}{15} \\ \frac{1}{15} & \frac{29}{180} \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 4 \end{bmatrix}}$$

$$p(x_1, x_2) = \frac{1}{12\pi} e^{-\frac{1}{2}\left[\frac{1}{5}(x_1-1)^2 + 2 \cdot \frac{1}{15}(x_1-1)(x_2-4) + \frac{29}{180}(x_2-4)^2\right]}$$

$$p(y_1, y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(y_1^2 + y_2^2)}$$

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Note: $$-\frac{1}{2}\,[x_1-1,\ x_2-4]\begin{bmatrix}\frac{1}{5} & \frac{1}{15}\\ \frac{1}{15} & \frac{29}{180}\end{bmatrix}\begin{bmatrix}x_1-1\\ x_2-4\end{bmatrix}$$

$$= -\frac{1}{2}\,[x_1-1,\ x_2-4]\begin{bmatrix}\frac{1}{5}(x_1-1)+\frac{1}{15}(x_2-4)\\ \frac{1}{15}(x_1-1)+\frac{29}{180}(x_2-4)\end{bmatrix}$$

$$= -\frac{1}{2}\Big[\frac{1}{5}(x_1-1)^2+\frac{1}{15}(x_1-1)(x_2-4) + \frac{1}{15}(x_1-1)(x_2-4)+\frac{29}{180}(x_2-4)^2\Big]$$

$$= -\frac{1}{2}\Big[\frac{1}{5}(x_1-1)^2+\frac{2}{15}(x_1-1)(x_2-4)+\frac{29}{180}(x_2-4)^2\Big]$$

$$= -\frac{1}{10}(x_1-1)^2-\frac{1}{15}(x_1-1)(x_2-4)-\frac{29}{360}(x_2-4)^2$$

$$= a_{11}x_1^2 + a_{10}x_1 + a_0 + a_{22}x_2^2 + a_{20}x_2^2 + a_{12}x_1x_2$$

(*quadratic in x_1 and x_2 with cross terms*)

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ex: Generate 2-dim gaussian dist. from 1-dim gaussian dist. using $p(x,y) = p_X(x)\, p(y|x)$.

$$p(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}$$

where $-1 \le \rho \le 1$ is correlation coefficient

Find $p_{Y|X=x}(y)$. We use $p_{Y|X=x}(y) = \dfrac{p(x,y)}{\int_{-\infty}^{\infty} p(x,y)\,dy}$.

$$\int_{-\infty}^{\infty} p(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}\,dy$$

Rewrite the exponent by completing the square:

$$x^2 - 2\rho xy + y^2 = (y-\rho x)^2 + \underbrace{x^2 - \rho^2 x^2}_{(1-\rho^2)x^2}$$

Also, we use $e^{a+b} = e^a \cdot e^b$:

$$\int_{-\infty}^{\infty} p(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(y-\rho x)^2/2(1-\rho^2)} \underbrace{e^{-\cancel{(1-\rho^2)}x^2/2\cancel{(1-\rho^2)}}}\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} e^{-(y-\rho x)^2/2\sigma^2}\,dy}_{\int \text{gaussian with } \mu_Y \equiv \rho x} e^{-x^2/2}$$

where we define $\sigma^2 \equiv 1-\rho^2$

We observe that $\int$ gaussian(with $\mu_Y = \rho x$) $= 1$.

$$\therefore \int_{-\infty}^{\infty} p(x,y)\,dy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$ standard gaussian i.e. $\mu = 0$, $\sigma^2 = 1$

$$\therefore p_{Y|X=x}(y) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2) + x^2/2}$$