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Jointly Distributed Random Variables

def: X and Y are jointly distributed random variables $\equiv$ when we perform two experiments with outcomes X and Y, we can describe the probability density (of having $X=x$ and $Y=y$ as the pair of outcomes) by using a function of x and y:

$$p(x,y) = \text{probability density for outcome } X=x \text{ and } Y=y$$

def: joint cumulative probability function $\equiv F(x,y)$

$$F(x,y) \equiv P\{x \le X \text{ and } y \le Y\}$$

tool:

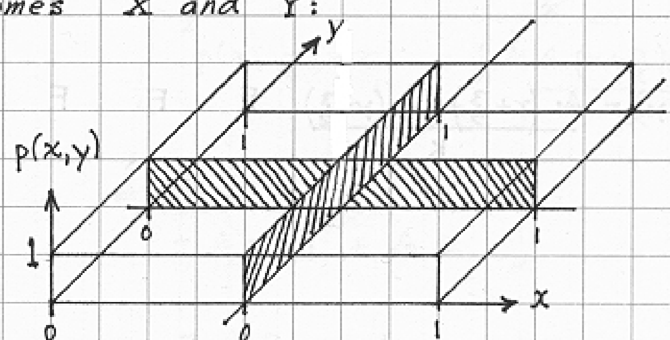
$$p(x,y) = \frac{d}{dx}\frac{d}{dy}F(x,y) = \frac{d}{dy}\frac{d}{dx}F(x,y)$$

and

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} p(x,y)\,dy\,dx = \int_{-\infty}^{y}\int_{-\infty}^{x} p(x,y)\,dx\,dy$$

ex: Suppose we perform two experiments that have no dependency on one another but are both uniformly distributed: $X \sim u(0,1)$, $Y \sim u(0,1)$.

Then $p(x,y)$ describes the probability of the pair of outcomes X and Y:



$p(x,y)$ is a unit cube, and its cross sections have the same shape as the distributions for X and Y. This is a result of X and Y being independent.

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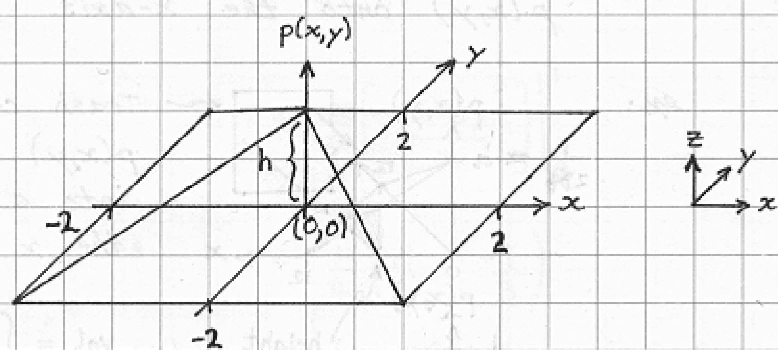
tool: The volume of $p(x,y)$ equals one for a valid probability density function:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = 1$$

Note: This corresponds to $\int_{-\infty}^{\infty} p(x)dx = 1$ for single random variable.

tool: $F(\infty,\infty) = 1$ (Since $P\{x \le \infty$ and $y \le \infty\} = 1$)

ex: Suppose $p(x,y)$ is pyramid-shaped. Find its height, h.



$$1 = \text{vol } p(x,y) = \int_0^h \text{area of square slice taken horizontally } dz$$

$$1 = \int_0^h \left(4\left(\frac{h-z}{h}\right)\right)^2 dz$$ Change var to $\ell = \frac{h-z}{h}$

Then $\frac{d\ell}{dz} = \frac{-1}{h}$ $\quad dz = \frac{-d\ell}{h}$

$z=0 \Rightarrow \ell = 1$

$z=h \Rightarrow \ell = 0$

$$1 = -\int_1^0 4\ell^2 \frac{d\ell}{h}$$

$$1 = \frac{4}{h}\int_0^1 \ell^2\,d\ell$$

$$1 = \frac{4}{h}\left.\frac{\ell^3}{3}\right|_0^1 = \frac{4}{3h}$$

$$\therefore\ h = \frac{4}{3}$$

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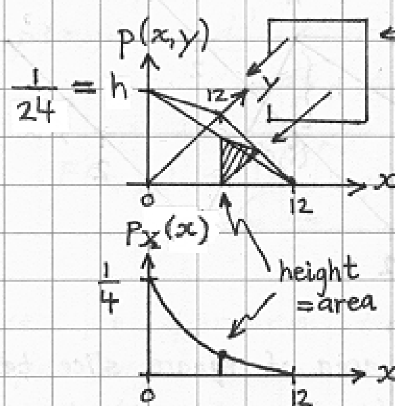
def: *marginal probability density function*, $p_X(x) \equiv$

$$p_X(x) \equiv \int_{-\infty}^{\infty} p(x,y)\, dy$$

Note: $p_X(x)$ is the probality density for X when we don't know the value of Y.

Note: $p_X(x)$ may be thought of as the density of matter we would get if we compacted $p(x,y)$ onto the x-axis.

ex:



Trash compactor squashes $p(x,y)$ in y direction into a pane sitting on the x axis.

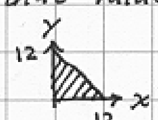
$$\text{vol} = \int_0^h \tfrac{1}{2}(144)\left(\frac{z}{h}\right)^2 dz = \frac{72}{h^2}\frac{z^3}{3}\Bigg|_0^h$$

$$\text{vol} = \frac{72h}{3} \Rightarrow h = \frac{1}{24}$$

$p_X(x)$ is quadratic

Height of pane on x axis is mass of slice in y direction for that value of x.

$$\text{Here, } p(x,y) = \begin{cases} \frac{1}{24} \cdot \frac{12-(x+y)}{12} & \text{on} \\ 0 & \text{otherwise} \end{cases}$$



$$p_X(x) = \int_{-\infty}^{\infty} p(x,y)\, dy = \int_0^{y=12-x} \frac{1}{24}\,\frac{12-(x+y)}{12}\, dy$$

$$= \frac{1}{24}\int_0^{12-x} \frac{12-x}{12} - \frac{y}{12}\, dy = \frac{1}{24}\left[\frac{(12-x)^2}{12} - \frac{y^2}{24}\Bigg|_0^{12-x}\right]$$

$$= \frac{(12-x)^2}{24^2} = \frac{1}{4}\left(1 - x/12\right)^2 \quad \text{for } x \text{ in } [0,12]$$

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tool: $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$ since $\int_{-\infty}^{\infty} p_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = 1$

This means $p_X(x)$ is a valid probability density function, and we don't have to scale $p_X(x)$ up or down to make $p_X(x)$ valid.

ex: Check previous example.

$$\int_{-\infty}^{\infty} p_X(x)\,dx = \int_0^{12} \frac{1}{4}\left(1-\frac{x}{12}\right)^2 dx$$

$$'' = \frac{1}{4}\int_0^1 (1-y)^2\, 12\,dy \qquad y = \frac{x}{12}$$

$$'' = \frac{12}{4}\left[-\frac{(1-y)^3}{3}\right]\Bigg|_0^1$$

$$'' = \frac{12}{4}\left(0 - \frac{-1}{3}\right)$$

$$\int_{-\infty}^{\infty} p_X(x)\,dx = 1 \quad \checkmark$$

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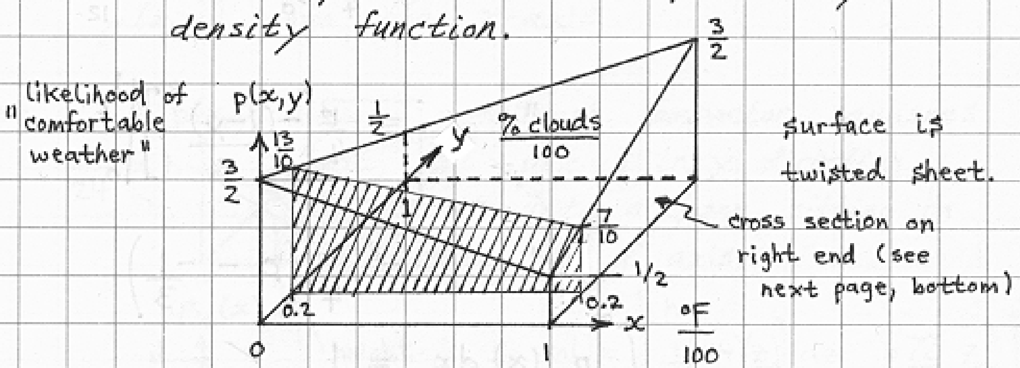
def: conditional probability density function, $p_{X|Y=y}(x)$

$$p_{X|Y=y}(x) \equiv \frac{p(x,y)}{p_Y(y)} = \text{probability density for } x \text{ given a particular value of } Y = y$$

plug in particular value of y, e.g. $y = \frac{1}{2}$

Note: The denominator, $p_Y(y)$, is a constant that normalizes the conditional density function so the area under the conditional density function equals one. This is necessary for a "valid probability density function."

ex:



What is the probability density of "comfortable weather" versus temperature given the cloud cover is 20%?

$$p_{X|Y=0.2}(x) = \frac{p(x,y=0.2)}{p_Y(y=0.2)} = \frac{p(x,y=0.2)}{\int_{-\infty}^{\infty} p(x,y=0.2)\,dx}$$

Note that $p(x,y=0.2)$ is the cross-hatched area in the above plot of $p(x,y)$.

We normalize the probability density function for $p_{X|Y=0.2}(x)$ by dividing by the area of the cross section: $\int_{-\infty}^{\infty} p(x,y=0.2)\,dx = p_Y(y=0.2)$

ex: (cont.) The following equation describes the cross section of $p(x,y)$ at $y=0.2$:

$$p(x, y=0.2) = \begin{cases} \frac{13}{10} - \frac{6}{10}x & x \text{ in } [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then
$$p_Y(y=0.2) = \int_0^1 \frac{13}{10} - \frac{6}{10}x \, dx$$
$$= \frac{13}{10}x \Big|_0^1 - \frac{6}{10}\frac{x^2}{2}\Big|_0^1$$
$$= \frac{13}{10} - \frac{6}{10}\frac{1}{2}$$
$$= 1$$

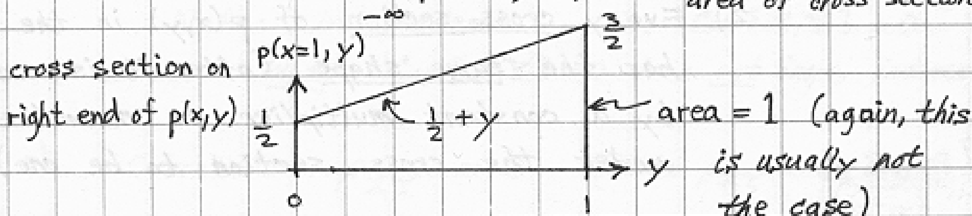
So, by serendipity, our normalizing factor is just 1. This is <u>not</u> usually the case.

def: $$p_{Y|X=x}(y) \equiv \frac{p(x,y)}{p_X(x)} = \frac{p(x,y)}{\int_{-\infty}^{\infty} p(x,y)\,dy}$$

Note: definition just reverses roles of x and y compared with definition of $p_{X|Y=y}(x)$.

ex: Find $p_{Y|X=1}(y)$ for "comfortable weather" $p(x,y)$, above.

soln: $$p_{Y|X=1}(y) = \frac{p(x=1,y)}{\int_{-\infty}^{\infty} p(x=1,y)\,dy} = \frac{\text{cross section on right end of } p(x,y)}{\text{area of cross section}}$$



$$p_{Y|X=1}(y) = \begin{cases} \frac{\frac{1}{2}+y}{1} = \frac{1}{2}+y & y \text{ in } [0,1] \\ 0 & \text{otherwise} \end{cases}$$