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Independent Random Variables

def: X and Y are independent $\equiv$ outcome X is not affected by outcome Y

$\equiv$ knowing outcome Y doesn't change the probability density function for X

$\equiv$ knowing outcome Y doesn't help us improve our guess of what X is likely to be

$$\equiv p_{X|Y=y}(x) = p_X(x)$$

$$\equiv p_{Y|X=x}(y) = p_Y(y)$$

tool: If X and Y are independent, we can split $p(x,y)$ into a part dependent only on x and a part dependent only on y:

$$X \text{ and } Y \text{ independent} \Rightarrow p(x,y) = p_X(x)\,p_Y(y)$$

~~tool~~: X and Y independent implies the following results:

1) $p(x,y) = p_X(x)\,p_Y(y)$

2) Every cross section of $p(x,y)$ in the x direction has the same shape (after it is scaled up or down by a constant multiplier to normalize the area under the cross section to be one).

3) Same as (2) but for cross sections in the y direction.

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results for X and Y independent continued: (see later notes for defs)

4) $$E\{XY\} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x\, p_X(x)\, dx \cdot \int_{-\infty}^{\infty} y\, p_Y(y)\, dy$$

$$= E\{X\}E\{Y\}$$

$$= \mu_X \mu_Y$$

(We can't assume this result without independence.)

5) $$\mathrm{Cov}\{X,Y\} \equiv E\{(X-\mu_X)(Y-\mu_Y)\}$$

$$= E\{X-\mu_X\}E\{Y-\mu_Y\}$$

$$= (E\{X\}-\mu_X)(E\{Y\}-\mu_Y)$$

$$= 0 \cdot 0$$

$$= 0$$

This says there is no average DC power in the variations of X and Y relative to their mean values. In other words, they don't tend to vary in the same direction. Instead, they are as likely vary in the opposite direction as in the same direction, intuitively speaking.

Note: independence implies $\mathrm{Cov}\{X,Y\} = 0$
<u>but</u> $\mathrm{Cov}\{X,Y\} = 0$ does <u>not</u> imply independence.

6) Correlation $\rho_{XY} = 0$ (since $\rho_{XY} = \dfrac{\mathrm{Cov}\{X,Y\}}{\sqrt{\sigma_X^2 \sigma_Y^2}}$)

Note: As with $\mathrm{Cov}\{X,Y\}$, independent $\Rightarrow \rho_{XY}=0$
but $\rho_{XY}=0 \not\Rightarrow$ independent.

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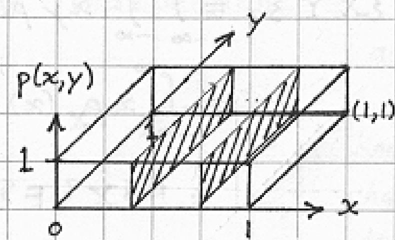
ex: Determine which of the following $p(x,y)$ represent independent X and Y:



Every cross section in the y direction has the same shape (even without normalization that would scale the cross section up or down).

$\therefore X$ and Y are independent

We can write $p(x,y) = p_X(x)\, p_Y(y)$

$$\text{where } p_X(x) = \begin{cases} 1 & x \text{ in } [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$p_Y(y) = \begin{cases} 1 & y \text{ in } [0,1] \\ 0 & \text{otherwise} \end{cases}$$

In other words, $p(x,y)$ is the product of a uniform distribution in the x direction and a uniform distribution in the y direction.

Note: Some cross sections (e.g., for $x = \frac{3}{2}$) are actually zero, but we are allowed to consider this as whatever cross section we wish. For the above example, we see that the cross section in the y direction at $x = \frac{3}{2}$ is $p_Y(y)$ multiplied by $p_X(\frac{3}{2}) = 0$. So we still consider the cross section to be $p_Y(y)$.

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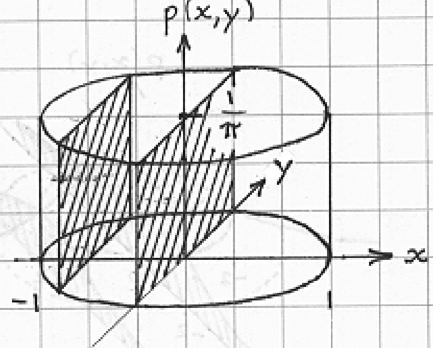
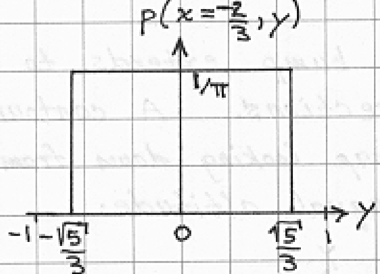
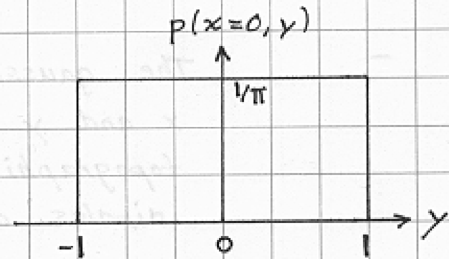
ex: (cont.) (Determine if $p(x,y)$ represents independent X and Y)



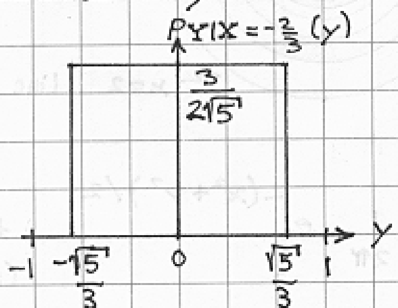
water tank
x = lattitude
y = longitude
$p(x,y)$ = prob density of finding water molecule at (x,y)

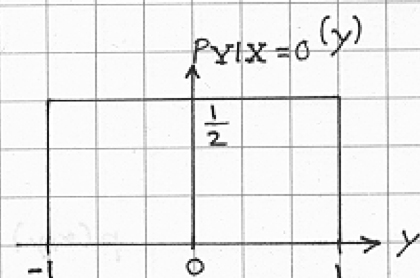
Cross sections in y direction:





Normalize curves (divide by their area) to get area under each curve = 1. (We are actually finding conditional probability density functions this way.)





These normalized cross sections do not have the same shape.

$\therefore$ X and Y are not independent

Can we have circularly symmetric $p(x,y)$, (See next example), with X and Y independent?

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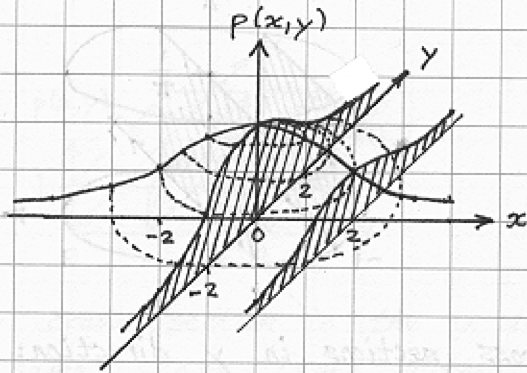
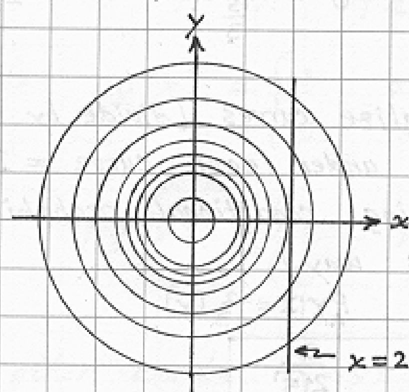
ex: (cont.) (Determine if $p(x,y)$ represents independent X and Y)



2-dimensional gaussian = round bump

circularly symmetric

The gaussian bump extends to $\pm\infty$ in both x and y directions. A contour plot (like topographic map looking down from the top) has circles of equal altitude:



top view of $p(x,y)$

line of cross section

$$p(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$ for all x,y (entire x,y plane)

Consider cross sections at $x=0$ and $x=2$:

$$p(0,y) = \frac{1}{2\pi} e^{-y^2/2} \qquad p(2,y) = \frac{1}{2\pi} e^{(-4-y^2)/2}$$

$$p(2,y) = \frac{1}{2\pi} e^{-4} e^{-y^2/2}$$

They both have shape $\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ after normalization.

$$p(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = p_X(x)p_Y(y)$$ X, Y are independent.