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Covariance

tool: Covariance measures the power in the product of the variations of two random variables relative to their means. In other words, if we consider $(X-\mu_X)\cdot(Y-\mu_Y)$, then the covariance of X and Y is average value of this product.

def: $\text{Cov}\{X,Y\} \equiv$ Covariance of X and Y $= E\{(X-\mu_X)(Y-\mu_Y)\}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\,p(x,y)\,dy\,dx$$

ex: In signal processing applications we often take the product of two signals. Two examples are demodulators [used in radios to shift a signal from a high frequency to a low frequency (e.g., audio) range] and matched filters [used to detect whether a signal has a particular waveform].

If we consider μ_X to be an actual signal, then $X-\mu_X$ measures the noise added to the signal, (sunspots!). Similarly, $Y-\mu_Y$ is the noise present on top of the actual signal μ_Y.

What is the power of the noise in the product of the signals on average?
Answer: $\text{Cov}\{X,Y\}$

What is the power for the product of the actual signals?
Answer: $\mu_X\cdot\mu_Y$ i.e., $E\{X\}\cdot E\{Y\}$

This relates to our next tool

tool: $\text{Cov}\{X,Y\} = E\{XY\} - E\{X\}E\{Y\} = E\{XY\} - \mu_X\mu_Y$

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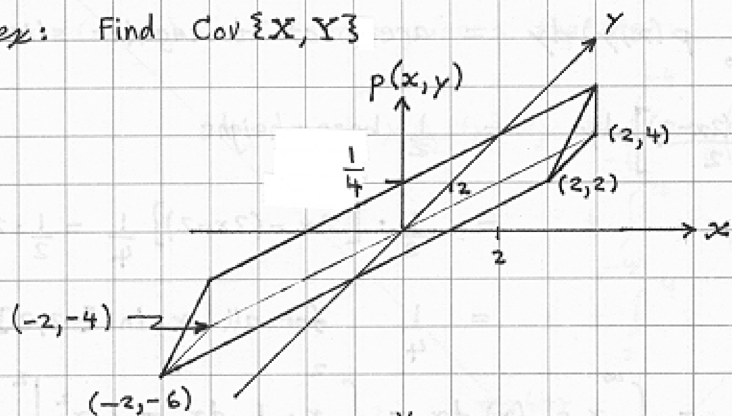
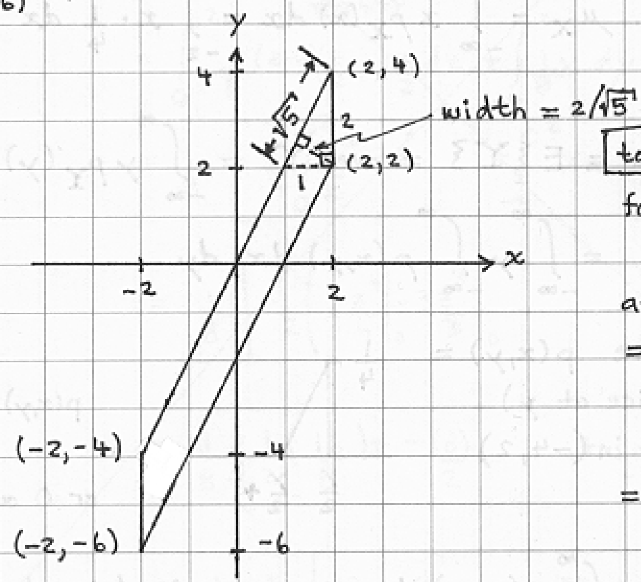
ex: Find Cov{X, Y}



prism shaped $p(x,y)$



$\boxed{\text{top view}}$ showing footprint of prism

area of footprint
= length of long side
· width of prism

$= \sqrt{4^2+8^2} \cdot \frac{2}{\sqrt{5}} = 8$

volume of prism = area of footprint · height · $\frac{1}{2}$

" $= 8 \cdot \frac{1}{4} \cdot \frac{1}{2} = 1$

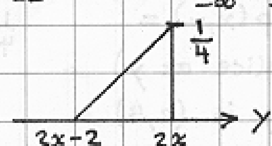
" $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx$

$$\text{Cov}\{X,Y\} = E\{(X-\mu_X)(Y-\mu_Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\,p(x,y)\cdot dy\,dx$$

$\mu_X = E\{X\} = 0$ from symmetry of footprint for $\pm x$

or use $\mu_X = \int_{-\infty}^{\infty} x\, p_X(x)\,dx = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p(x,y)\,dy\,dx$

where $p(x,y) =$ (slice at x)



$p(x,y) = \frac{1}{4}\,\frac{y-(2x-2)}{2}$

or 0 outside $[2x-2, 2x]$

ex: (cont.) $p_X(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$ = area of triangular slice

or $\int_{2x-2}^{2x} \frac{1}{4}\left[\frac{y-(2x-2)}{2}\right] dy$ $= \frac{1}{2}$ base $\cdot$ height

$= \frac{1}{2} \cdot [2x-(2x-2)]\,\frac{1}{4} = \frac{1}{2}\cdot 2 \cdot \frac{1}{4}$

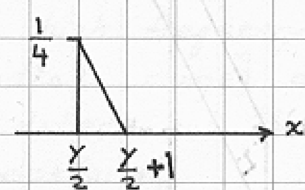
$= \frac{1}{4}$ for all x in $[-2,2]$

$\therefore \mu_X = \int_{-\infty}^{\infty} x\, p_X(x)\,dx = \int_{-2}^{2} x \cdot \frac{1}{4}\,dx = \frac{1}{4}\frac{x^2}{2}\Big|_{-2}^{2} = 0$

$\mu_Y = E\{Y\} = \text{ave } Y = \int_{-\infty}^{\infty} y\, p_Y(y)\,dy$

$'' \quad = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} p(x,y)\,dx\,dy$

where $p(x,y) =$ (slice at y) y in $(-4,2)$



$p(x,y) = \frac{1}{4}\left[\frac{y}{2}+1-x\right]$

or 0 outside $\left[\frac{y}{2}, \frac{y}{2}+1\right]$

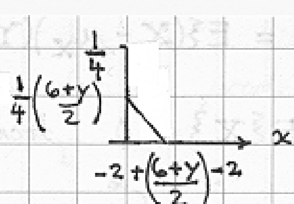
$p_Y(y) = \int_{-\infty}^{\infty} p(x,y)\,dx$ = area of triangle

or $\int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{1}{4}\left(\frac{y}{2}+1-x\right)dx$ $= \frac{1}{2}$ base $\cdot$ height

$= \frac{1}{2}\cdot 1 \cdot \frac{1}{4} = \frac{1}{8}$ all y in $[-4,2]$

Now for the end pieces:

$p(x,y) =$ (slice at y) y in $(-6,-4)$

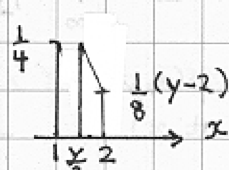


$p_Y(y)$ = area of triangle

$= \frac{1}{2}\left(\frac{6+y}{2}\right) \cdot \frac{1}{4}\left(\frac{6+y}{2}\right)$

$= \frac{1}{8}\left(\frac{6+y}{2}\right)^2$

$p(x,y) =$ (slice at y) y in $(2,4)$



$p_Y(y)$ = area of wedge

$= \frac{1}{4}\left(2-\frac{y}{2}\right) - \frac{1}{2}\left[\frac{1}{4}-\frac{1}{8}(y-2)\right]\left(2-\frac{y}{2}\right)$

ex: (cont.) After simplification we get area of wedge =

$$\frac{1}{8}\left(2-\frac{y}{2}\right)\frac{y}{2} = p_Y(y) \quad \text{for } y \text{ in } (2,4)$$

Tie the pieces together:

$$\mu_Y = \int_{-\infty}^{\infty} y\, p_Y(y)\, dy = \int_{-6}^{-4} y \cdot \frac{1}{8}\left(\frac{6+y}{2}\right)^2 dy$$

$$+ \int_{-4}^{2} y \cdot \frac{1}{8}\, dy$$

$$+ \int_{2}^{4} y \cdot \frac{1}{8}\left(2-\frac{y}{2}\right)\frac{y}{2}\, dy$$

let $z = 6+y \qquad y = z-6 \qquad y=-6 \Rightarrow z=0 \qquad y=-4 \Rightarrow z=2$

$$\mu_Y = \frac{1}{8}\left[\int_0^2 (z-6)\left(\frac{z}{2}\right)^2 dz + \int_{-4}^{2} y\, dy + \int_2^4 \frac{y^2}{2}\left(2-\frac{y}{2}\right) dy\right]$$

$$= \frac{1}{8}\left[\int_0^2 \frac{1}{4}\left(z^3-6z^2\right) dz + \int_{-4}^{2} y\, dy + \int_2^4 \frac{2y^2}{2} - \frac{y^3}{4}\, dy\right]$$

$$= \frac{1}{8}\left[\frac{1}{4}\frac{z^4}{4}\Big|_0^2 - \frac{6}{4}\frac{z^3}{3}\Big|_0^2 + \frac{y^2}{2}\Big|_{-4}^{2} + \frac{y^3}{3}\Big|_2^4 - \frac{1}{4}\frac{y^4}{4}\Big|_2^4\right]$$

$$= \frac{1}{8}\left[\frac{1}{16}\cdot 16 - \frac{6}{4}\,\frac{8}{3} - \frac{12}{2} + \frac{1}{3}\,56 - \frac{1}{16}\,240\right]$$

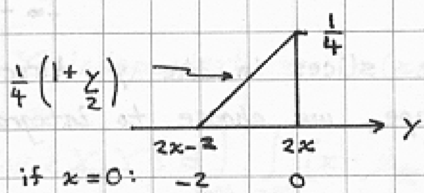
$$= \frac{1}{8}\left[\frac{12}{12} - \frac{48}{12} - \frac{72}{12} + \frac{224}{12} - \frac{180}{12}\right]$$

$$= \frac{1}{8}\,\frac{-64}{12}$$

$$= -\frac{2}{3}$$

This was the <u>hard way</u> to find μ_Y. A much <u>easier way</u> is to observe that every slice of $p(x,y)$ in the y direction has the same shape. We can take the mean of this shape, and then we find the mean position of these points in the y direction.

ex: (cont.) The cross section in the y direction is



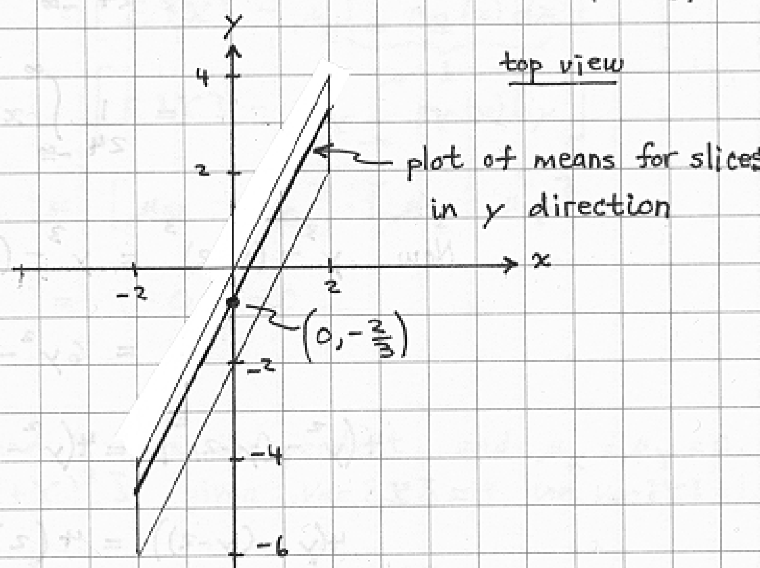
Scale the slice so its area $= 1$ and find mean val:

$$\mu_{x=0} = \int_{-2}^{0} y \cdot \frac{1}{4} \cdot 4\left(1+\frac{y}{2}\right) dy = \int_{-2}^{0} y\left(1+\frac{y}{2}\right) dy$$

$$= \int_{-2}^{0} y + \frac{y^2}{2}\, dy = \frac{y^2}{2}\Bigg|_{-2}^{0} + \frac{1}{2}\frac{y^3}{3}\Bigg|_{-2}^{0}$$

$$= -\frac{4}{2} + \frac{8}{6} = -\frac{4}{6} = -\frac{2}{3}$$

Now we observe that the plot of these mean points is a line that is symmetrical in the y direction — above and below the point at $\left(0, -\frac{2}{3}\right)$:



From symmetry, the average position of the means in the y direction is the same as the mean for $x=0$, namely $-\frac{2}{3}$.

$\therefore\ \mu_Y = -\frac{2}{3}$ This is the easy way, and more reliable.

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ex: (cont.) So now we have $Cov\{X,Y\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-0)(y--\frac{2}{3})p(x,y)\,dy\,dx.$

Since the slices in the y direction all have the same shape, we choose to integrate in the y direction first.

From the previous page (top) we have:

$$p(x,y) = \begin{cases} \frac{1}{4}\left(1+\frac{y}{2}\right) & \text{for } y \text{ in } [2x-2,2x] \\ 0 & y \text{ outside } [2x-2,2x] \end{cases}$$

Thus, $Cov\{X,Y\} = \int_{-2}^{2} x \int_{2x-2}^{2x}\left(y+\frac{2}{3}\right)\frac{1}{4}\left(1+\frac{y}{2}\right)dy\,dx$

$$= \int_{-2}^{2} x\,\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\int_{2x-2}^{2x}(3y+2)(2+y)\,dy\,dx$$

$$= \frac{1}{24}\int_{-2}^{2} x\int_{2x-2}^{2x} 3y^2+8y+4\,dy\,dx$$

$$= \frac{1}{24}\int_{-2}^{2} x\left(\cancel{3}\frac{y^3}{\cancel{3}}+8\frac{y^2}{2}+4y\Big|_{2x-2}^{2x}\right)dx$$

Now $y^3-(y-2)^3 = y^3-(y^3-6y^2+6y-8)$

$$= 6y^2-6y+8$$

$$4\left(y^2-(y-2)^2\right) = 4\left(y^2-(y^2-4y+4)\right) = 4(4y-4)$$

$$4(y-(y-2)) = 4(2) = 8$$

Thus, $Cov\{X,Y\} = \frac{1}{24}\int_{-2}^{2} x\left(6y^2-6y+\cancel{8}+16y-\cancel{16}+\cancel{8}\Big|_{y=2x}\right)dx$

$$'' = \frac{1}{24}\int_{-2}^{2} x\left(24x^2+20x\right)dx = \frac{1\cdot 2}{24}\int_{0}^{2}20x^2dx$$

$$'' = \frac{40}{24}\frac{x^3}{3}\Big|_0^2 = \frac{40\cdot 8}{24\cdot 3} = \frac{40}{9}$$ Finally!

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tool: If X and Y are independent, then $Cov\{X,Y\} = 0$

pf: X and Y independent $\Rightarrow$ $p(x,y) = p_X(x)\,p_Y(y)$

Then $Cov\{X,Y\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\,p(x,y)\,dy\,dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\,p_X(x)\,p_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}(x-\mu_X)\,p_X(x)\,dx \cdot \int_{-\infty}^{\infty}(y-\mu_Y)\,p_Y(y)\,dy$$

$$= \left[\int_{-\infty}^{\infty} x\,p_X(x)\,dx - \int_{-\infty}^{\infty}\mu_X\,p_X(x)\,dx\right]$$

$$\cdot\left[\int_{-\infty}^{\infty} y\,p_Y(y)\,dy - \int_{-\infty}^{\infty}\mu_Y\,p_Y(y)\,dy\right]$$

$$= \left[E\{X\} - \mu_X\underbrace{\int_{-\infty}^{\infty} p_X(x)\,dx}_{1}\right]\left[E\{Y\} - \mu_Y\int_{-\infty}^{\infty} p_Y(y)\,dy\right]$$

$$= [\mu_X - \mu_X]\cdot[\mu_Y - \mu_Y]$$

$$= 0 \cdot 0 = 0 .$$

ex: Suppose X and Y are independent and $\mu_X = \mu_Y = 0$. Find $E\{(X+Y)^2\}$ given $Var\{X\} = 4$ and $Var\{Y\} = 1$.

soln: $E\{(X+Y)^2\} = E\{X^2 + 2XY + Y^2\}$

$$= E\{X^2\} + 2E\{XY\} + E\{Y^2\}$$

$$= Var\{X\} + 2\,Cov\{X,Y\} + Var\{Y\}$$

since μ_X and $\mu_Y = 0$

$$= Var\{X\} + 2\cdot 0 + Var\{Y\} = 4+1 = 5$$