Counting Techniques • P{poker hands}, P{dice combinations}

~~tool~~: # possible outcomes from sequence of expts = size of sample space, $\mathcal{S}$

= product of possible outcomes for each expt

$= n_1 \cdot n_2 \cdot \ldots \cdot n_K$

$n_1 \equiv$ # possible outcomes for 1^{st} expt • e.g. 1^{st} die

$n_2 \equiv$ " " " " 2^{nd} " " 2^{nd} die

$\vdots$

ex: We roll three dice of different colors. What is the # of possible outcomes?

soln: $6 \cdot 6 \cdot 6$ = 216

6 ← # possible outcomes 3^{rd} die

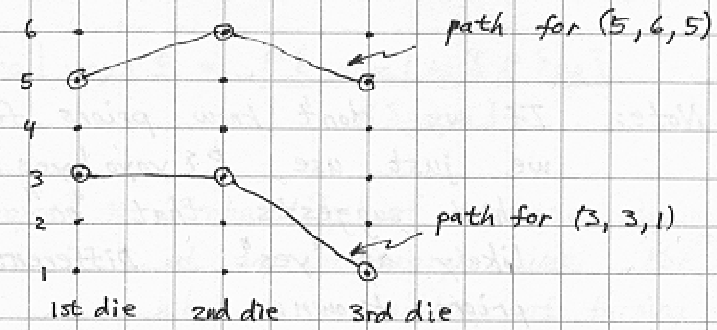
6 ← # possible outcomes 2^{nd} die

6 ← # possible outcomes 1^{st} die

• # Possible outcomes gets very large very quickly

⇒ Curse of dimensionality

• # possible outcomes = # paths thru chart of outcomes per expt



Two possible paths shown. Total # possible paths = 6 possible choices for 1^{st} die

• 6 " " " 2^{nd} "

• 6 " " " 3^{rd} "

• Paths equally likely for fair die.

• When outcomes all equally likely, we can use permutations and combinations to compute probabilities of outcomes.

noťn: $n! \equiv 1 \cdot 2 \cdot \ldots \cdot n \equiv n$ factorial

• Note: $0! = 1$

def: $P_k^n \equiv$ # of permutations of n objects taken k at a time (where order matters and we are sampling without replacement)

$$= n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-k+1) = \frac{n!}{(n-k)!}$$

ex: Drawing names out of a hat for a lottery. Don't put names back in if they get drawn.

Suppose we have 60 students and we draw 3 names in succession from the hat. How many possible outcomes if we keep track of who is drawn when?

1^{st} name	60 possible	names	
• 2^{nd} "	59	"	"
• 3^{rd} "	58	"	"

Total # possible outcomes $= 60 \cdot 59 \cdot 58 = 205{,}320.$

$P\{$ one particular sequence of 3 names $\} = \dfrac{1}{205{,}320}$

def: $C_k^n \equiv \binom{n}{k} =$ # of possible subsets of k objects from a set of n objects (where order does <u>not</u> matter and we are sampling without replacement)

$$= \frac{n!}{(n-k)!}$$

ex: How many sets of 3 names from hat if order they are drawn in is ignored.

tool: $C_k^n = \dfrac{P_k^n}{k!} = \dfrac{\text{\# of permutations of } n \text{ objects taken } k \text{ at a time}}{\text{\# permutations of those } k \text{ objects (taken } k \text{ at a time)}}$

ex: Consider drawing from hat with 60 names in it.

Suppose we want to know the probability that Abe, Becky, and Cathy win.

They could be drawn in $3! = 1\cdot 2\cdot 3 = 6$ ways.

3 possible names in 1st slot.

Then only two names left for 2nd slot. Can't use name twice.

Only one name left for third slot.

1st draw	2nd draw	3rd draw
Abe	Becky	Cathy
	Cathy	Becky
Becky	Abe	Cathy
	Cathy	Abe
Cathy	Abe	Becky
	Becky	Abe

P{ Abe, Becky, Cathy drawn from hat in any order }

$$= 1 \Big/ \left(\# \text{ sets of 3 names we can draw} = C^n_k = \frac{P^n_k}{k!} = \frac{P^{60}_3}{3!} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{60!}{57!\,3!} = \frac{60\cdot 59\cdot 58}{3\cdot 2} = 34{,}220 \right)$$

$$= \frac{1}{34{,}220}$$

- Like picking a trifecta in horse racing if we do consider order. This reduces probability of winning by factor of $k! = 6$.

ex: How many ways are there to put 5 x's and 4 o's on a tic-tac-toe board?

soln: Find # ways to place 5 x's in 9 squares (leaving 4 squares blank). Since all x's look the same, order of x's doesn't matter.

e.g.

x	x	
	x	x
	x	

$\therefore$ # ways to place 5 x's in 9 squares $= \binom{9}{5}$

$$= \frac{9!}{(9-5)!\,5!} = \frac{9\cdot \overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot \cancel{5}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 126$$

Now find # ways to place 4 o's in 4 remaining squares. Hmm... clearly only one way! But let's see if the math works. The answer should be $\binom{4}{4} = \frac{4!}{(4-4)!\,4!} = \frac{1}{0!} = \frac{1}{1} = 1$ ✓

So our final answer is $126 \cdot 1 = 126$.