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Correlation

def: $\text{Corr}\{X,Y\}$ or $\rho_{XY} \equiv \dfrac{\text{Cov}\{X,Y\}}{\sqrt{\sigma_X^2\sigma_Y^2}}$

tool: The correlation measures the power in the noise that is common to both signals X and Y, (this is the $\text{Cov}\{X,Y\}$ in the numerator), and then it normalizes the result by dividing by the total power in the noise for signals X and Y.

Consequently, $-1 \le \rho_{XY} \le 1$.

tool: Correlation ρ_{XY} has the following properties:

If $Y = X$ then $\rho_{XY} = 1$

$Y = -X$ $\rho_{XY} = -1$

X, Y independent $\rho_{XY} = 0$

Note: For $p(x,y)$ with reflective symmetry about the x or y axes, or if we choose μ_X and μ_Y appropriately, we can have $\text{Cov}\{X,Y\}=0$ even though X and Y are not independent.

Since $\text{Cov}\{X,Y\}=0$ implies $\rho_{XY}=0$ we have:

$\rho_{XY}=0$ does not imply X,Y independent

ex: Find ρ_{XY} for the prism-shaped function presented as an example in the section on covariance

sol'n: $\text{Cov}\{X,Y\} = \frac{40}{9}$ Now find σ_X^2 and σ_Y^2.

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ex: (cont.) $\sigma_X^2 = E\{X^2\} - \mu_X^2 = \int_{-\infty}^{\infty} x^2 p_X(x)\,dx - \mu_X^2$

From prism shaped $p(x,y)$ we have (see "covariance" notes)

$$p_X(x) = \begin{cases} \frac{1}{4} & x \text{ in } [-2,2] \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_X = 0$.

Thus, $\sigma_X^2 = \int_{-2}^{2} x^2 \cdot \frac{1}{4}\,dx = \frac{1}{4} \frac{x^3}{3}\Big|_{-2}^{2} = \frac{1}{6} \cdot 8 = \frac{4}{3}$

$$\sigma_Y^2 = E\{Y^2\} - \mu_Y^2 = \int_{-\infty}^{\infty} y^2 p_Y(y)\,dy - \mu_Y^2$$

From prism example we have

$$p_Y(y) = \begin{cases} \frac{1}{8}\left(\frac{6+y}{2}\right)^2 & y \text{ in } [-6,-4] \\ \frac{1}{8} & y \text{ in } [-4,2] \\ \frac{1}{8}\left(2-\frac{y}{2}\right)\frac{y}{2} & y \text{ in } [2,4] \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_Y = -\frac{2}{3}$.

Thus, $\sigma_Y^2 = \int_{-6}^{-4} y^2 \frac{1}{8}\left(\frac{6+y}{2}\right)^2 dy$

$+ \int_{-4}^{2} y^2 \frac{1}{8}\,dy$

$+ \int_{2}^{4} y^2 \frac{1}{8}\left(2-\frac{y}{2}\right)\frac{y}{2}\,dy$

Calculate each piece:

$$\int_{-6}^{-4} y^2 \frac{1}{8}\left(\frac{6+y}{2}\right)^2 dy = \int_0^2 (z-6)^2 \frac{1}{8}\left(\frac{z}{2}\right)^2 dz \qquad \begin{matrix} z=y+6 \\ y=z-6 \end{matrix}$$

$$= \frac{1}{8\cdot 4}\int_0^2 (z^2-12z+36)z^2\,dz$$

$$= \frac{1}{32}\left(\frac{z^5}{5} - 12\frac{z^4}{4} + 36\frac{z^3}{3}\Big|_0^2\right) = \frac{1}{32}\left(\frac{32}{5} - \overbrace{3\cdot 16 + 12\cdot 8}^{+48}\right)$$

$$= 272/32 = 17/2$$

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ex: (cont.) $\int_{-4}^{2} y^2 \cdot \frac{1}{8}\, dy = \frac{1}{8} \frac{y^3}{3}\Big|_{-4}^{2} = \frac{1}{24}(8+64) = 3$

$$\int_2^4 y^2 \frac{1}{8}\left(2-\frac{y}{2}\right)\frac{y}{2}\, dy = \frac{1}{8\cdot 4}\int_2^4 y^2 (4-y)\, y\, dy$$

$$= \frac{1}{32}\int_2^4 y^2(4y-y^2)\, dy = \frac{1}{32}\int_2^4 4y^3 - y^4\, dy$$

$$= \frac{1}{32}\left(\cancel{4}\frac{y^4}{\cancel{4}} - \frac{y^5}{5}\Big|_2^4\right)$$

$$= \frac{1}{32}\left[(256-16) - \frac{(1024-32)}{5}\right]$$

$$= \frac{1}{32}\left[240 - \frac{992}{5}\right] = \frac{1}{32}\,\frac{1200-992}{5}$$

$$= \frac{208}{32}$$

$$= \frac{13}{2}$$

$$\therefore\ \sigma_Y^2 = \frac{17}{2} + 3 + \frac{13}{2} = 18$$

$$\rho_{XY} \equiv \frac{Cov\{X,Y\}}{\sqrt{\sigma_X^2\ \sigma_Y^2}} = \frac{40/9}{\sqrt{\frac{4}{3}\cdot 18}} = \frac{40/9}{\sqrt{24}} = \frac{20}{9\sqrt{6}}$$

$\rho_{XY} = 0.907$ • Note that $-1 \le \rho_{XY} \le 1$ is verified.

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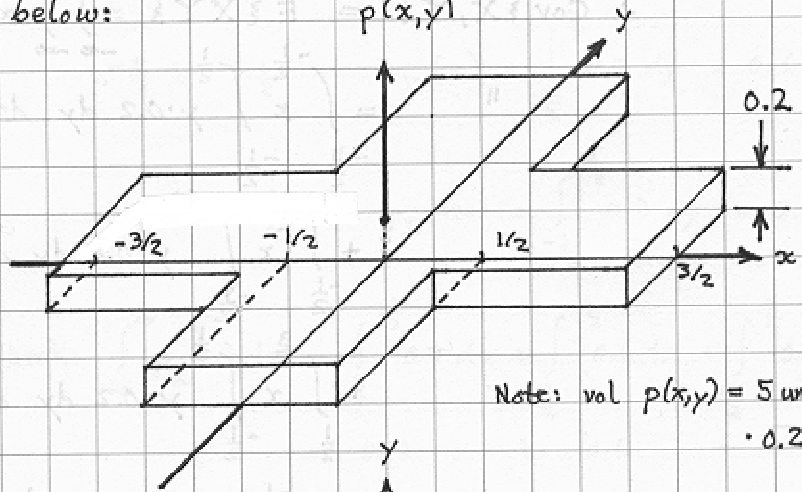
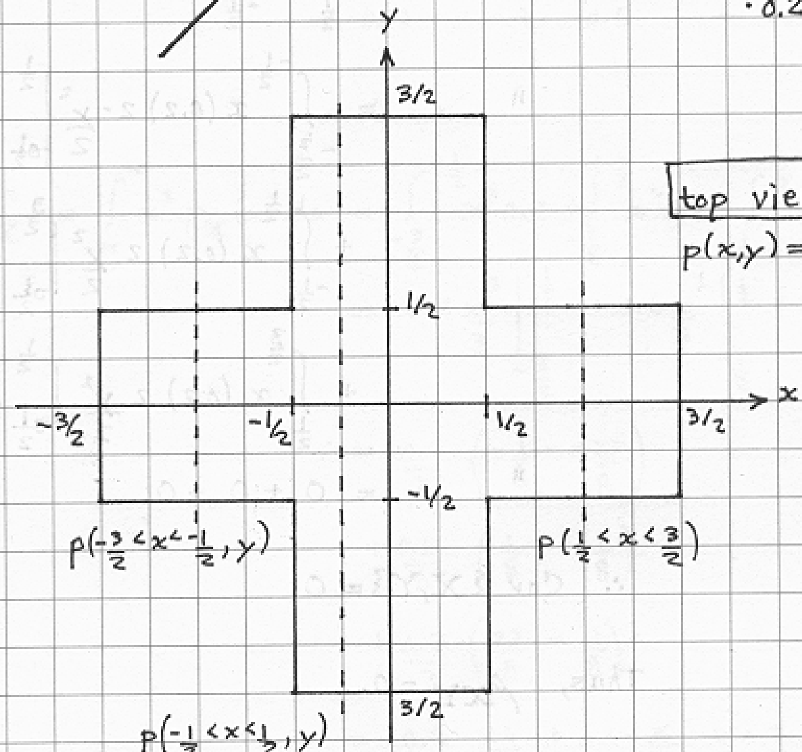
ex: Find ρ_{XY} for the joint probability density function shown below:

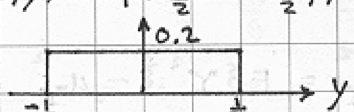


Note: vol $p(x,y)$ = 5 unit squares $\cdot\, 0.2 = 1$ ✓

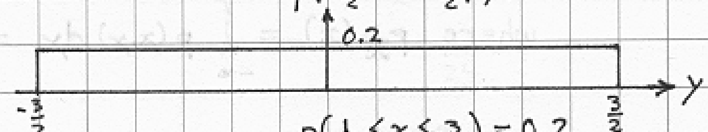


top view
$p(x,y) = 0.2$

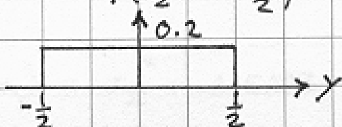
∴ cross sections of $p(x,y)$: $p(-\frac{3}{2} < x < -\frac{1}{2}, y) = 0.2$



$p(-\frac{1}{2} < x < \frac{1}{2}, y) = 0.2$



$p(\frac{1}{2} < x < \frac{3}{2}) = 0.2$



ex: (cont.) From symmetry, $\mu_X = \mu_Y = 0$.

$$\therefore \text{Cov}\{X,Y\} = E\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y)\, dy\, dx$$

$$= \int_{-\frac{3}{2}}^{-\frac{1}{2}} x \int_{-\frac{1}{2}}^{\frac{1}{2}} y \cdot 0.2\, dy\, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x \int_{-\frac{3}{2}}^{3/2} y \cdot 0.2\, dy\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} x \int_{-\frac{1}{2}}^{\frac{1}{2}} y \cdot 0.2\, dy\, dx$$

$$= \int_{-\frac{3}{2}}^{-\frac{1}{2}} x(0.2) \left.\frac{y^2}{2}\right|_{-\frac{1}{2}}^{\frac{1}{2}} dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x(0.2) \left.\frac{y^2}{2}\right|_{-\frac{3}{2}}^{\frac{3}{2}} dx + \int_{\frac{1}{2}}^{\frac{3}{2}} x(0.2) \left.\frac{y^2}{2}\right|_{-\frac{1}{2}}^{\frac{1}{2}} dx$$

$$= 0 + 0 + 0$$

$\therefore$ $\text{Cov}\{X,Y\} = 0$

Thus, $\rho_{XY} = 0$.

For the sake of illustration, we compute σ_X^2:

$$\sigma_X^2 = E\{X^2\} - \mu_X^2 = E\{X^2\} = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx$$

where $p_X(x) = \int_{-\infty}^{\infty} p(x,y)\, dy$ = area under cross section shown at bottom of previous page for $p(x,y)$ [with appropriate choice of curve given x].

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ex: (cont.) For $-\frac{3}{2} < x < -\frac{1}{2}$, $p_X(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 0.2\,dy = 0.2$

$-\frac{1}{2} < x < \frac{1}{2}$ $\quad p_X(x) = \int_{-\frac{3}{2}}^{\frac{3}{2}} 0.2\,dy = 0.6$

$\frac{1}{2} < x < \frac{3}{2}$ $\quad p_X(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 0.2\,dy = 0.2$

Check: $\int_{-\infty}^{\infty} p_X(x)\,dx = \int_{-\frac{3}{2}}^{-\frac{1}{2}} 0.2\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 0.6\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} 0.2\,dx$

$= 0.2 + 0.6 + 0.2$

$= 1$ ✓

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 p_X(x)\,dx = \int_{-\frac{3}{2}}^{-\frac{1}{2}} x^2 0.2 + \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 0.6 + \int_{\frac{1}{2}}^{\frac{3}{2}} x^2 0.2$$

$$= 0.2\frac{x^3}{3}\Big|_{-\frac{3}{2}}^{-\frac{1}{2}} + 0.6\frac{x^3}{3}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + 0.2\frac{x^3}{3}\Big|_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{0.2}{3}\left(-\frac{1}{8} - -\frac{27}{8}\right)$$

$$+ \frac{0.6}{3}\left(\frac{1}{8} - -\frac{1}{8}\right)$$

$$+ \frac{0.2}{3}\left(\frac{27}{8} - \frac{1}{8}\right)$$

$$= \frac{0.2}{3}\frac{26}{8} + \frac{0.6}{\cancel{2}}\frac{\cancel{2}}{8} + \frac{0.2}{3}\frac{26}{8}$$

$$= \frac{0.2}{3}\frac{26}{8} + \frac{0.6}{3}\frac{3}{8} + \frac{0.2}{3}\frac{26}{8}$$

$$= \frac{0.2}{24}\left(26 + 3\cdot 3 + 26\right)$$

$$= 0.2 \cdot \frac{61}{24}$$

$$= 0.5083$$

By symmetry, $\sigma_Y^2 = \sigma_X^2 = 0.5083$ as well.

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ex: Calculate $\rho_{X^2, 3X^2}$.

soln: Define $Y \equiv X^2$. Then $\rho_{X^2, 3X^2} = \rho_{Y, 3Y}$.

$$\rho_{Y,3Y} = \frac{\text{Cov}\{Y, 3Y\}}{\sqrt{\sigma_Y^2 \, \sigma_{3Y}^2}}$$

$$\text{Cov}\{Y, 3Y\} = E\{Y \cdot 3Y\} - \mu_Y \mu_{3Y}$$

$$'' \quad = 3E\{Y^2\} - \mu_Y \, 3\mu_Y$$

$$'' \quad = 3\left(E\{Y^2\} - \mu_Y^2\right)$$

$$'' \quad = 3\sigma_Y^2$$

$$\sigma_{3Y}^2 = E\{3Y \cdot 3Y\} - \mu_{3Y}^2$$

$$'' \quad = 9E\{Y^2\} - 3^2\mu_Y^2$$

$$'' \quad = 9\left(E\{Y^2\} - \mu_Y^2\right)$$

$$'' \quad = 9\sigma_Y^2$$

$$\therefore \rho_{Y,3Y} = \frac{3\sigma_Y^2}{\sqrt{\sigma_Y^2 \, 9\sigma_Y^2}} \qquad \frac{3\sigma_Y^2}{3\sigma_Y^2}$$

$$\rho_{Y,3Y} = 1$$

$$\therefore \rho_{X^2, 3X^2} = 1$$

Note: Since $3X^2$ is a scaled version of X^2 and ρ normalizes for energy, we get $\rho = 1$. (Deterministic relationship between X^2 and $3X^2$.)