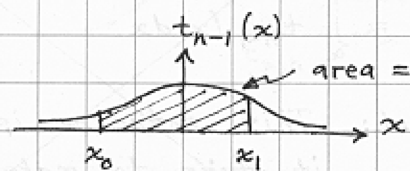
Estimating the distribution of μ from $\bar{X}$ and S

Earlier we showed $\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

where t_{n-1} is t (or student's) distribution with $n-1$ degrees of freedom.

Using the idea that probability corresponds to the area under the probability density function, we have

$$P\left\{x_0 \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le x_1\right\} = \int_{x_0}^{x_1} t_{n-1}(x)\,dx$$

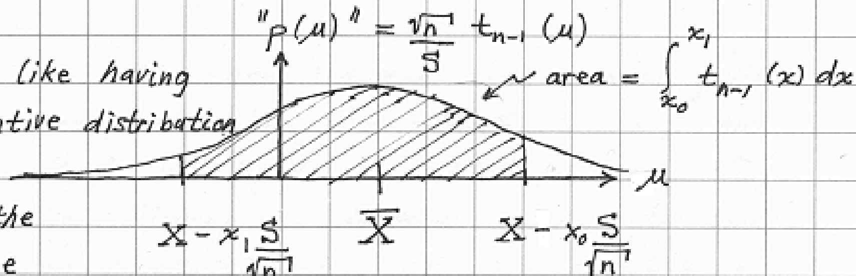


After a few steps of algebra on the inequality inside the probability function, i.e. inside $P\{\ \}$, we have

$$P\left\{\bar{X} - x_1\frac{S}{\sqrt{n}} \le \mu \le \bar{X} - x_0\frac{S}{\sqrt{n}}\right\} = \int_{x_0}^{x_1} t_{n-1}(x)\,dx$$

This remarkable result tells us the probability that the true mean μ lies in the interval $\left[\bar{X} - x_1\dfrac{S}{\sqrt{n}},\ X - x_0\dfrac{S}{\sqrt{n}}\right]$ is $\int_{x_0}^{x_1} t_{n-1}(x)\,dx$.

This is like having a cumulative distribution for μ. Taking the derivative



gives us something like a probility density function for μ.

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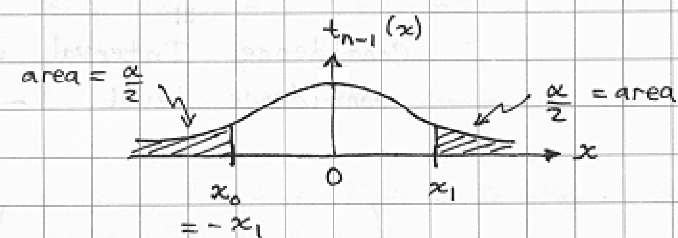
So we have $"p(\mu)" = \frac{\sqrt{n}}{S}\, t_{n-1}(\mu - \overline{X})$

Comment: We use quote marks because what we really have is the probability density for getting an $\overline{X}$ and S such that μ lies within a certain distance from $\overline{X}$ (where the distance is scaled by S).

After all, μ is a fixed value. So we are really determining whether $\overline{X}$ is close to μ.

For confidence intervals we choose x_0 and x_1 such that $\int_{-\infty}^{x_0} t_{n-1}(x)\,dx = \frac{\alpha}{2} = \int_{x_1}^{\infty} t_{n-1}(x)\,dx$:

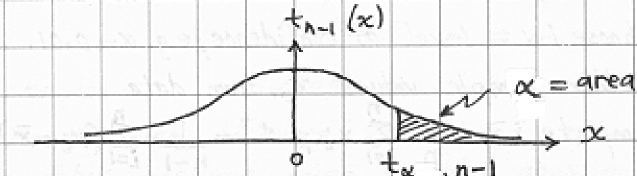


Since $t_{n-1}(x)$ is symmetric $x_0 = -x_1$.

not'n: $t_{\frac{\alpha}{2}, n-1} \equiv x_1$ such that $\int_{x_1}^{\infty} t_{n-1}(x)\,dx = \frac{\alpha}{2}$

In other words, $\int_{t_{\frac{\alpha}{2}, n-1}}^{\infty} t_{n-1}(x)\,dx = \frac{\alpha}{2}$.

Also, $\int_{t_{\alpha, n-1}}^{\infty} t_{n-1}(x)\,dx = \alpha$.



Confidence Interval

It follows that

$$P\left\{ \overline{X} - t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}} \le \mu \le \overline{X} + t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}} \right\} = 1 - \alpha$$

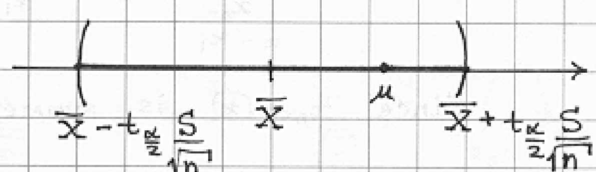
or

$$P\left\{ \mu \in \left[\overline{X} - t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}, \overline{X} + t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \right] \right\} = 1 - \alpha$$

or

$$P\left\{ \mu \notin \underbrace{\left[X - t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}, \overline{X} + t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \right]}_{\equiv} \right\} = \alpha$$

Confidence Interval with confidence level $1-\alpha$



The probability that this *interval actually contains the true* μ *is* $1-\alpha$.

$$P\{\text{confidence interval contains } \mu\} = 1 - \alpha$$

Computing Confidence Interval:

1. Choose $1-\alpha \equiv$ level of confidence, e.g. $\alpha = 0.01$.
2. Gather sample values x_i.
3. Compute $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$.
4. Use table to find $t_{\frac{\alpha}{2}, n-1}$.
5. Plug in $\overline{x}$, s, $t_{\frac{\alpha}{2}, n-1}$ and n to find confidence interval.

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ex: An engineer wants to determine the average noise level, μ, for signals sent (by a cell phone one mile away) to a base station

It turns out the noise level doesn't depend on how far away the cell phone is, (although the strength of the signal from the cell phone does).

Thus, the engineer measures the signal at the base station when no cell phones are transmitting.

Realizing that the value of μ cannot be determined exactly from data, the engineer decides to calculate a confidence interval that has a 99% probability of containing μ.

The engineer collects 121 samples with a digital 'scope. From these samples, she finds $\bar{x} = 0.3V$ and $s^2 = 900\mu V^2$.

$\alpha = 1\%$ or 0.01 for 99% confidence level.

Using a table (such as Table III in Hayter "Prob. and Stats. for Eng. and Sci." p 884), $t_{\frac{\alpha}{2},n} = t_{\frac{0.01}{2},\,121-1} = t_{0.005,\,120}$

$t_{0.005,\,120} = 2.617$

$$t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}} = \frac{2.617\cdot\sqrt{900\mu V^2}}{\sqrt{121}} = \frac{2.617\cdot 30mV}{11} = 7.14\ mV$$

Our confidence interval is $\bar{x} \pm 7.14mV = 300mV \pm 7.14mV$ or $[292.86\ mV, 307.14\ mV]$.

There is a 99% probability that the actual μ is between 292.86 mV and 307.14 mV.

The engineer concludes the base station receiver has a DC bias between 292mV and 308mV (probably).
Note: this assumes noise samples gaussian and independent.