ex: Full house in Yahtzee game (5 dice) is three of a kind and two of a kind.
Find $P\{\text{full house (and not 5 of a kind)}\}$.

sol'n: $P\{\text{full house}\} = \dfrac{\#\text{ full houses possible}}{\#\text{ outcomes possible with 5 dice}}$

$\#\text{ outcomes possible with 5 dice} = \underbrace{6\cdot 6\cdot 6\cdot 6\cdot 6}_{\text{5 dice, 6 faces}} = 6^5$

Now we count how many full houses possible.
• Fasten seat belts...

Consider a full house:

die	1	2	3	4	5
color	red	green	white	blue	gold

We can imagine the dice are different colors and rolled in order. Once we roll a die, we set it aside, and we don't roll it again. This is like sampling without replacement.

If we do have a full house, we can identify each die as belonging to the pair or to the triplet. These do not overlap (can't be in pair e.g. of 1's and triplet of 2's at the same time), and they account for all five dice.

We start by counting how many ways there are to get 3 dice the same out of 5. ~~Six~~ ~~there~~ are 6 different #'s we could have (i.e. 3 ones, 3 twos, ..., or 3 sixes) we have:

$\#\text{ways to get 3 of kind} = 6(\#\text{ways to distribute (or choose) the 3 dice out of the 5 dice})$

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We can think of the # of ways to choose the 3 dice out of the 5 dice as sampling without replacement: we start with 5 dice and randomly pick one. We set it aside and randomly pick one die from the remaining 4. We set that aside and pick the 3^{rd} die at random from the remaining 3. We now have our triplet.

Since it is sampling without replacement, we have:

$$\# \text{ ways to choose 3 dice out of 5} = C_3^5 = \binom{5}{3} = \frac{5!}{(5-3)!\,3!}$$

$$= \frac{5\cdot \cancel{4}^{2}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot 1\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 10.$$

Thus, # ways to get 3 of a kind $= 6\cdot 10 = 60.$

Now we have just the pair left to worry about. After accounting for the triplet, we only have two dice out of five left. We calculate the number of ways to get a pair for two dice. But we also have to avoid having the same number for the pair as for the triplet (or we'll have 5 of a kind). So the # on the dice for the pair can only have 5 possible values instead of six.

ways to get pair $= 5$

Thus, the total # full houses $= 60\cdot 5 = 300.$

$$P\{\text{full house (not 5 of kind)}\} = \frac{300}{6^5} = 0.0386$$

or roughly 1 in 26.

So 1 out of 26 first shakes should be a full house in Yahtzee.

· It took me 14 shakes. Just lucky, I guess.

23 Jan 03
Neil E Cotter

ex: Yahtzee game (5 dice). Shake the 5 dice, repeat 44 times. Find: P{3 full houses (3 of a kind and pair but not 5 of a kind) in 44 shakes}

Earlier we calculated P{full house} $= \frac{300}{6^5} = \frac{50}{6^4} = \frac{25}{648}$ for one throw.

The outcomes for different throws are independent because the outcome of one throw doesn't affect the outcome of another throw.

Thus, the probability of a particular sequence of outcomes — namely 3 full houses followed by 41 non full houses — is given by the product of the probabilities of those outcomes:

P{3 full houses followed by 41 non full houses}

$$= P\{\text{full house}\}^3 \; P\{\text{non full house}\}^{41}$$

$$= \left(\frac{25}{648}\right)^3 \left(1 - \frac{25}{648}\right)^{41} = 1.14 \cdot 10^{-5}$$

We multiply this probabity (which is the same for any sequence of hands that has 3 full houses and 41 non full houses, regardless of the order in which they occur) by the # ways to have 3 full houses in 44 shakes.

In other words, we multiply by $C_3^{44} \equiv \binom{44}{3} = \frac{44 \cdot 43 \cdot \cancel{42}^{7}}{\cancel{3 \cdot 2} \cdot 1}$

$$= 13244$$

Note: We don't care which full house appears first, second, or third. Thus, we use C_3^{44} rather than P_3^{44}.

P{3 full houses in 44 shakes} $= 1.14 \cdot 10^{-5} \cdot 13244 = 0.15$ or 15%

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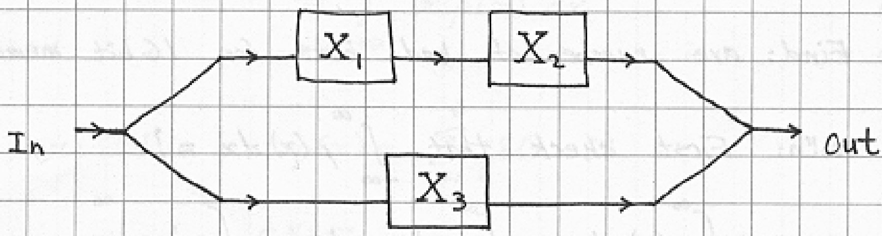
ex: Counting paths. Water flows in on left.
Valve closed = 0. Three valves, 1, 2, and 3.
" open = 1



Will water flow out on right?

Probabilities for valves: $P\{X_1=0\} = 0.8$ $\therefore P\{X_1=1\}=0.2$
$P\{X_2=0\} = 0.7$ $\therefore P\{X_2=1\}=0.3$
$P\{X_3=0\} = 0.6$ $\therefore P\{X_3=1\}=0.4$

First we consider the solution when X_1, X_2, and X_3 are independent.

soln: We can enumerate all possible outcomes and sum the probabilities of solutions that allow water to flow.

Outcome	Flow	$P\{X_3\}$	· $P\{X_2\}$	· $P\{X_1\}$	$= P\{\text{Outcome}\}$
0 0 0	0	0.6	0.7	0.8	= 0.336
0 0 1	0	0.6	0.7	0.2	= 0.084
0 1 0	0	0.6	0.3	0.8	= 0.144
0 1 1	1	0.6	0.3	0.2	= 0.036
1 0 0	1	0.4	0.7	0.8	= 0.224
1 0 1	1	0.4	0.7	0.2	= 0.056
1 1 0	1	0.4	0.3	0.8	= 0.096
1 1 1	1	0.4	0.3	0.2	= 0.024
$X_3 X_2 X_1$				Tot	= 1

Sum of outcomes that allow flow
$0.036 + 0.224 + 0.056 + 0.096 + 0.024 = 0.436$

This was the hard way.

ex: (cont.) Another way to solve this problem is to calculate $P\{\text{not flow}\}$ and use $P\{\text{flow}\} = 1 - P\{\text{not flow}\}$.

$$P\{\text{not flow}\} = P\{\text{both paths blocked}\}$$
$$= P\{\text{top path blocked} \cap \text{bottom path blocked}\}$$
$$= P\{X_1 = 0 \cup X_2 = 0\} \underbrace{P\{X_3 = 0\}}_{0.6} \text{ independent}$$

Use this trick again to calculate $P\{X_1 = 0 \cup X_2 = 0\}$.

$$P\{X_1 = 0 \cup X_2 = 0\} = 1 - P\{X_1 = 1 \cap X_2 = 1\}$$
$$" \quad = 1 - P\{X_1 = 1\} \cdot P\{X_2 = 1\} \ \underline{\text{ind}}$$
$$" \quad = 1 - 0.2 \cdot 0.3$$
$$" \quad = 1 - 0.06$$
$$" \quad = 0.94$$

$$\therefore P\{\text{not flow}\} = 0.94 \cdot 0.6 = 0.564$$

$$\therefore P\{\text{flow}\} = 1 - P\{\text{not flow}\} = 0.436$$

Another way to solve this problem is to find a partion of the set of outcomes that simplifies the calculation of the total probability.

From the chart of outcomes on the previous page, we see that we can partition the outcomes into three convenient events:

$A_1 \equiv \{X_3 = 1\}$ (flow)

$A_2 \equiv \{(X_3, X_2, X_1) = (0, 1, 1)\}$ (flow)

$A_3 \equiv \mathcal{S} - (A_1 \cup A_2)$ i.e., remainder of outcomes (no flow)

We sum the probabilities of partition elements that result in flow:

$$P\{\text{flow}\} = P\{A_1\} + P\{A_2\} = P\{X_3 = 1\} + P\{(0,1,1)\} = 0.4 + 0.036$$
$$= 0.436 \checkmark$$

ex: (cont.) Now we consider the solution when X_3 is independent of X_1 and X_2 but X_1 and X_2 are <u>dependent</u> random variables.

$P\{X_3=0\} = 0.8$ $P\{X_2=0\} = 0.7$ $P\{X_1=0\} = 0.6$

Same as before.

Dependent probabilities:

$P\{(X_1, X_2)\}$:

	$X_1 =$ 0	1	row sum $= P\{X_2\}$ ↓
$X_2 =$ 0	0.64	0.06	0.7
1	0.16	0.14	0.3
col sum $= P\{X_1\}$ →	0.8	0.2	

We have $P\{(X_1, X_2)\} \equiv P\{X_1 \cap X_2\} = P\{X_2 | X_1\} P\{X_1\}$

Thus, $P\{X_2 | X_1\} = \dfrac{P\{X_1 \cap X_2\}}{P\{X_1\}}$

$P\{0|0\} = \dfrac{0.64}{0.8} = 0.8$ $P\{1|0\} = \dfrac{0.16}{0.8} = 0.2$

$P\{0|1\} = \dfrac{0.06}{0.2} = 0.3$ $P\{1|1\} = \dfrac{0.14}{0.2} = 0.7$

Curiously, $P\{X_2|1\} \stackrel{?}{=} P\{X_2\}$ so $X_2 = 0$ or 1 and $X_1 = 1$ are independent events? No. $P\{X_2|1\} = 1 - P\{X_2\}$. Probability <u>flipped</u>.

But $P\{X_2|0\} \neq P\{X_2\}$, so the variables are dependent.

The dependency affects the calculation of probabilities of outcomes.

ex: (cont.) If we calculate $P\{(0,1,0)\}$, for example, we have

$$P\{(0,1,0)\} = P\{X_3=0\} \cdot P\{X_2=1 \mid X_1=0\} P\{X_1=0\}$$
$$= 0.6 \cdot 0.2 \cdot 0.8$$
$$= 0.096 \quad \text{previously it was } 0.144$$

Q. Does the dependency of X_1 and X_2 change our answer for $P\{\text{flow}\}$?

A. Yes. It turns out that if we use the partition $A_1 \equiv \{X_3 = 1\}$, $A_2 \equiv \{(0,1,1)\}$ and $A_3 \equiv \mathscr{S} - (A_1 \cup A_2)$ as before,

then $P\{A_1\} = P\{X_3 = 1\} = 0.4$ as before

and $P\{A_2\} = P\{(0,1,1)\} = P\{X_3=0\} P\{X_2=1 \mid X_1=1\} \cdot P\{X_1=1\}$

$$= 0.6 \cdot 0.7 \cdot 0.2 = 0.084 \quad \text{\underline{different}.}$$

$P\{\text{flow}\} = 0.4 + 0.084 = 0.484$ higher than before.

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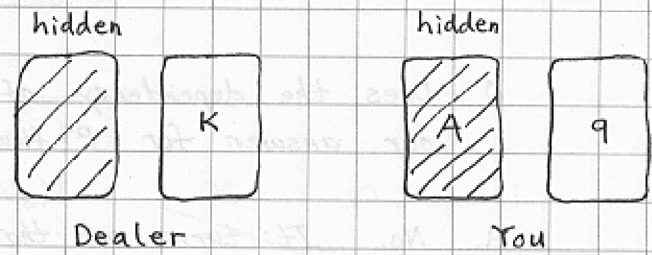
ex: Blackjack. Each player gets one card face down, and one card face up. Face cards count 10, an ace = 1 or 11 (your choice). Try to get total = 21.

Consider this situation:



Find: $P\{\text{Dealer has Ace hidden} \mid \text{Dealer has 9 showing and You have Ace 9}\}$

$$\equiv P\{\text{Dealer } A \mid \text{Dealer } 9 \ \& \ \underbrace{(A, 9)}_{\text{You}}\}$$

soln: $P\{\text{Dealer } A \mid \text{Dealer } 9 \ \& \ (A, 9)\}$

$$= \frac{\#\text{ Aces left when dealer 9 and your A,9 removed}}{\#\text{ cards left in deck}}$$

$$= \frac{4-1}{52-3} = \frac{3}{49} < \frac{1}{13} = \frac{3}{39} \text{ for fresh deck}$$

Odds of dealer blackjack reduced by ≈ 20%.

ex: Playing cards. Find P{3 or more aces in 5-card hand}.

soln: P{≥3 Aces in 5 cards}

= P{Ace, Ace, Ace, not Ace, not Ace} · # ways to select 3 cards out of 5

+ P{Ace, Ace, Ace, Ace, not Ace} · # ways to select 4 cards out of 5

Note: Since there are only four Aces, we can't get five.

$$P\{\ 3 \text{ Aces in 5 cards}\} = \left(\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{\cancel{48}}{49}\cdot\frac{47}{\cancel{48}}\right)\cdot C^5_3$$

$$\text{"} \quad = \frac{1}{13}\cdot\frac{1}{17}\cdot\frac{1}{25}\cdot\frac{47}{49}\cdot\left(\frac{5\cdot\cancel{4}^{2}}{\cancel{2}\cdot 1} = 10\right)$$

$$P\{4 \text{ Aces in 5 cards}\} = \frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{1}{49}\cdot\frac{48}{48}\cdot C^5_4$$

$$\text{"} \quad = \frac{1}{13}\cdot\frac{1}{17}\cdot\frac{1}{25}\cdot\frac{1}{49}\cdot\left(\frac{5}{1} = 5\right)$$

$$\therefore P\{\geq 3 \text{ Aces in 5 cards}\} = \frac{1}{13}\cdot\frac{1}{17}\cdot\frac{1}{25}\cdot\frac{47\cdot 10 + 1\cdot 5}{49}$$

$$\text{"} \quad = 1.75\cdot 10^{-3}$$

Roughly once in 500 hands, (i.e. $\approx 2\cdot 10^{-3}$).