ECE 3530 Chebyshev's Inequality

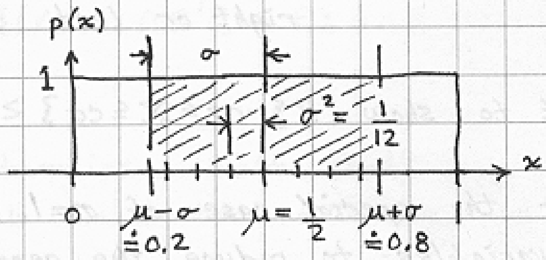
~~tool~~: Chebyshev's Inequality $\equiv P\{\mu - c\sigma \le X \le \mu + c\sigma\} \ge 1 - \frac{1}{c^2}$

where $\mu \equiv$ mean

$\sigma \equiv \sqrt{\text{variance}}$

$c \equiv$ how many σ from mean, $c \ge 1$ (req.)

ex:



$$\sigma^2 = \frac{1}{12} \qquad \sigma = \frac{1}{2\sqrt{3}} = 0.29 \approx 0.3$$

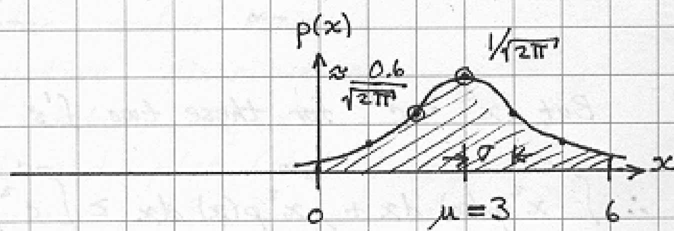
Here $c = 1$. $P\{\mu - c\sigma \le X \le \mu + c\sigma\} \ge 1 - \frac{1}{c^2}$

or $P\{0.2 \le X \le 0.8\} \ge 1 - \frac{1}{1} = 0$

- Hmm... doesn't help us much in this case. It's much more useful for distributions with long tails.
- Chebyshev tells us how much probability can be in tails.

ex: gaussian (i.e., normal) distribution $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

Suppose $\mu = 3$ and $\sigma^2 = 1 \Rightarrow \sigma = 1$



Find: a lower bound for the probability that the experiment outcome will be within 3σ of the mean $\mu = 3$, i.e. between 0 and 6 here.

soln: Chebyshev's Inequality says:

$$P\{\mu - 3\sigma \le X \le \mu + 3\sigma\} \ge 1 - \frac{1}{3^2} \doteq 0.89 \approx 90\%$$

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pf: (Chebyshev's Inequality)

Assume $\mu = 0$. $\mu \neq 0$ just shifts distribution right or left

Now we want to show $P\{-c\sigma \leq X \leq c\sigma\} \geq 1 - \frac{1}{c^2}$, $c > 1$.

We'll consider the special case of $\sigma = 1$. (Use a change of variables to reduce the general case to this case, if necessary.)

$\sigma = 1$ and we want to show $P\{-c \leq X \leq c\} \geq 1 - \frac{1}{c^2}$, $c > 1$.

Suppose the converse: $P\{-c \leq X \leq c\} < 1 - \frac{1}{c^2}$, $c > 1$.

This means $\int_{-c}^{c} p(x)\,dx < 1 - \frac{1}{c^2}$

Then $\int_{-\infty}^{-c} p(x)\,dx + \int_{c}^{\infty} p(x)\,dx > \frac{1}{c^2}$,

since $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

Consider $\sigma^2 = 1 = \int_{-\infty}^{\infty} x^2 p(x)\,dx \geq \int_{-\infty}^{-c} x^2 p(x)\,dx + \int_{c}^{\infty} x^2 p(x)\,dx$

But $x^2 \geq c^2$ for these two $\int$'s

$$\therefore \int_{-\infty}^{-c} x^2 p(x)\,dx + \int_{c}^{\infty} x^2 p(x)\,dx \geq \int_{-\infty}^{-c} c^2 p(x)\,dx + \int_{c}^{\infty} c^2 p(x)\,dx$$

$$\geq c^2 \underbrace{\left(\int_{-\infty}^{-c} p(x)\,dx + \int_{c}^{\infty} p(x)\,dx \right)}_{> \frac{1}{c^2} \text{ from earlier}}$$

> 1

Then $\sigma^2 = 1 > 1$. Impossible. $\therefore$ Converse is false.

$\therefore$ Theorem is true.