Spring 2007 ECE 6962 Advanced topics in communication Homework 2

Assign date: 1/30/06 Due date: 2/13/04 (Tuesday) at the beginning of class.

1. *Get familiar with the Q function.* Bounds of the Q function:

$$Q(x) = \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$$

(a) For $x > 0$ show that the following upper and lower bounds hold for the Q function:

$$(1 - \frac{1}{x^2})\frac{e^{-x^2/2}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-x^2/2}}{x\sqrt{2\pi}}$$

Hint: for the upper bound, write the integrand as a product of $1/t$ and $te^{-t^2/2}$, use integration by parts, and bound. For the lower bound, integrated by parts once more and bound.

(b) As you know that the bit error probability for BPSK signaling in additive white Gaussian noise (AWGN) channel with power spectral density (PSD) $N_0/2$ (per real dimension) is given by

$$P_e = Q(\sqrt{\frac{2E_b}{N_0}})$$

where E_b is the bit energy. Plot the error probability P_e (on a log scale) versus signal-to-noise ration E_b/N_0 (in dB) using Matlab or Mathematica. Consider E_b/N_0 ranging from -5 dB to 15 dB. Also plot the bounds and compare.

2. *The importance of Gray labeling.* Consider a discrete time AWGN channel model with

$$y = x + w.$$

The transmitted signal x is chosen from a QPSK constellation and the noise w is complex Gaussian with distribution $\mathcal{CN}(0, N_0)$.

(a) Write down the ML decision rule and specify the decision region.

(b) Assume that Gray labeling (shown in Figure 1) is used. Derive a closed-form expression for the symbol error probability $P_{e,s}$ given the ML decision rule.

(c) Use your answer in (b) to derive the bit error probability $P_{e,b}$ for Gray labeling. Verify that it equals $Q(\frac{a}{\sqrt{N_0/2}})$, which is the bit error probability when the two bits are detected independently.

(d) Compute $P_{e,s}$ and $P_{e,b}$ for the non-Gray labeling shown in Figure 1. How does it compare with the results of Gray labeling?

3. *Binary detection in AWGN channel.* Given a continuous-time model where

$$y(t) = \begin{cases} s_1(t) + w(t) & \text{if "1" sent} \\ s_0(t) + w(t) & \text{if "0" sent} \end{cases}$$

Assume that all signals are real signals and the noise process $w(t)$ is white Gaussian noise with PSD $N_0/2$ (per real dimension).

(a) The short approach:

Let $\tilde{y}(t) = y(t) - s_0(t)$ and consider an equivalent channel model

$$\tilde{y}(t) = \begin{cases} s_1(t) - s_0(t) + w(t) & \text{if "1" sent} \\ w(t) & \text{if "0" sent} \end{cases}$$

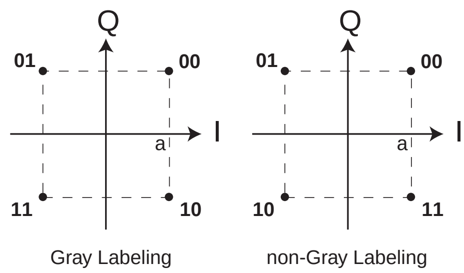

Figure 1: Problem 2.

Derive the ML decision rule and specify the decision threshold. Compute the error probability P_e and express it in terms of the Q-function.

(b) The longer approach: project the received signal $y(t)$ to the signal space spanned by $s_1(t)$ and $s_0(t)$. Define $Z_1 =< y(t), s_1(t) >$ and $Z_0 =< y(t), s_0(t) >$. It is shown in class that the ML rule can be expressed as

$$Z_1 - Z_0 \underset{\text{say } 0}{\overset{\text{say } 1}{\gtrless}} \frac{\|s_1\|^2 - \|s_0\|^2}{2}$$

Compute the error probability P_e. Verify that you obtain the same answers as in (a).

4. Consider the M-ary detection problem,

$$y(t) = x(t) + w(t)$$

where $x(t)$ equals one of the M possible signals $s_1(t)$, $s_2(t)$, $\cdots$, $s_M(t)$. Assume that all signals are real and the noise process $w(t)$ with PSD of $N_0/2$ (per real dimension). Also assume that $s_1(t)$, $s_2(t)$, $\cdots$, $s_M(t)$ have equal energy E_s.
(a) Convert the original problem to a discrete time problem and identify the ML decision rule.
(b) Assume further that the signals are orthogonal. Show that the error probability is given by

$$P_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [1 - Q(x)]^{M-1} e^{-(x-d)^2/2} dx, \qquad \text{where } d = \sqrt{\frac{2E_s}{N_0}}$$

5. *Noncoherent detection for a block fading channel* . Consider a discrete time block fading model:

$$\mathbf{y} = h\,\mathbf{x} + \mathbf{w},$$

where h is a complex random scalar representing Rayleigh fading. Assume that the receiver does not know h. Assume either x_A or x_B is transmitted, where $x_A, x_B \in \mathbb{C}^n$ and $\|x_A\|^2 = \|x_B\|^2$. The noise vector $\mathbf{w}$ has a complex Gaussian distribution $\mathcal{CN}(0, N_0 I_n)$.

Derive a ML rule for this detection problem. You don't need to compute the error probability P_e.

6. *Noncoherent detection for channel with unknown phase shift.* Consider a continuous-time model

$$y(t) = \begin{cases} s_1(t)e^{j\theta} + w(t) & \text{if "1" sent,} \\ s_0(t)e^{j\theta} + w(t) & \text{if "0" sent.} \end{cases}$$

where θ is some unknown channel phase shift that is uniformly distributed in $[0, 2\pi]$. Assume that all signals are complex, the noise process has PSD N_0, and $< s_1(t), s_0(t) >= 0$. Derive the ML detection rule and compute the probability of error.