ECE 6540, Lecture 23

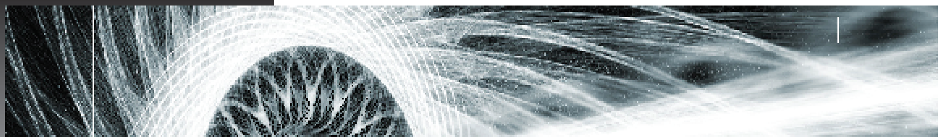
Karlin-Rubin Theorem Continued and
The Generalized Likelihood Ratio Test

News

- **Congratulations to Supreet for passing his qualifying exam!**

Advertisement

Will be teaching again in Spring 2017



ECE 6534: Advanced Digital Signal Processing II

Recent Trends in Signal Processing

Instructor:

Joel B. Harley
Assistant Professor of Electrical and Computer Engineering

Class Times:

Tuesday / Thursday
2:00 PM - 3:20 PM

Class Structure:

Part 1: Lectures
Part 2: Lectures, paper discussions, presentations, projects

Overview

We live in an era of information. Data permeates every part of our society and every technical field needs to process data. As a result, there have been many advances on how to effectively and efficiently process the data deluge. These advances make use of concepts from many disciplines, including linear algebra, representation theory, network theory, and machine learning. These methods provide many benefits, such as quicker data processing, more effective information extraction, and better data fusion from multiple sources. Throughout this course, we will explore many of these signal processing advances.

In the first part of the class, we will discuss the foundations of recent signal processing advances. In the second part of the class, student interests will dictate what we cover and with what depth. Topics of interest may include:

- **multidimensional signal processing**
- **wavelets**
- **optimization**
- **compressive sensing**
- **signal processing over graphs**
- **signal processing for machine learning**

Prerequisites

- **Graduate** standing or permission from instructor.
- Students from **all departments** welcome.
- <u>It is **not required or necessary** to have taken ECE 6533 (Advanced Digital Signal Processing I) prior to this course.</u>

News

- **Projects**
- **Posters**
- **Homework**
- **Exam**

Last Time

- **Composite Tests / Detectors**
- **One-sided tests**
- **Two-sided tests**
- **Karlin Rubin Theorem**
- **Extending the Karlin Rubin Theorem to vector data**

Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$
  - Then the uniformly most powerful (UMP) test is:

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x > \gamma \\ 0 & \text{if} \quad x < \gamma \end{cases}$$

$$\alpha = E[\phi(X); \theta_0, \mathcal{H}_0]$$

Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$
  - Then the uniformly most powerful (UMP) test is:

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x > \gamma \\ 0 & \text{if} \quad x < \gamma \end{cases}$$

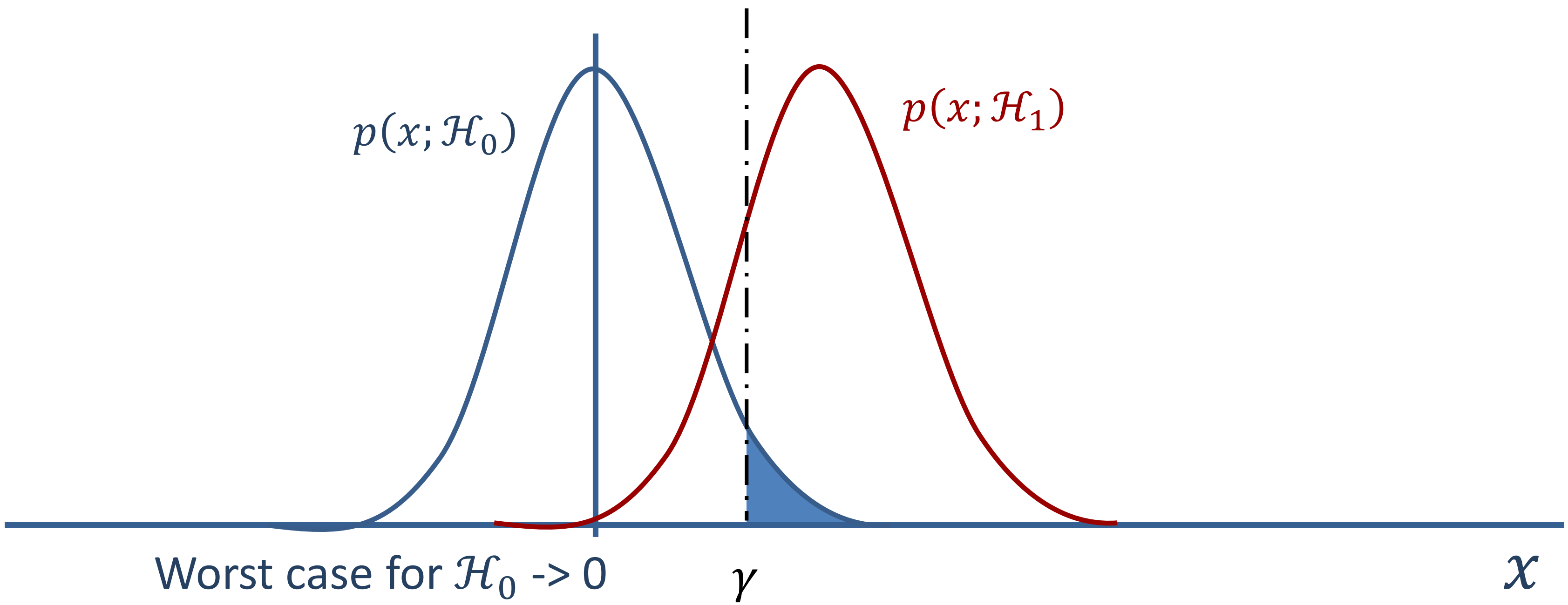
Note that we are fixing the worst-case false alarm rate

$$\alpha = E[\phi(X); \theta_0, \mathcal{H}_0]$$

Hence, the threshold does not depend on $\mathcal{H}_1$

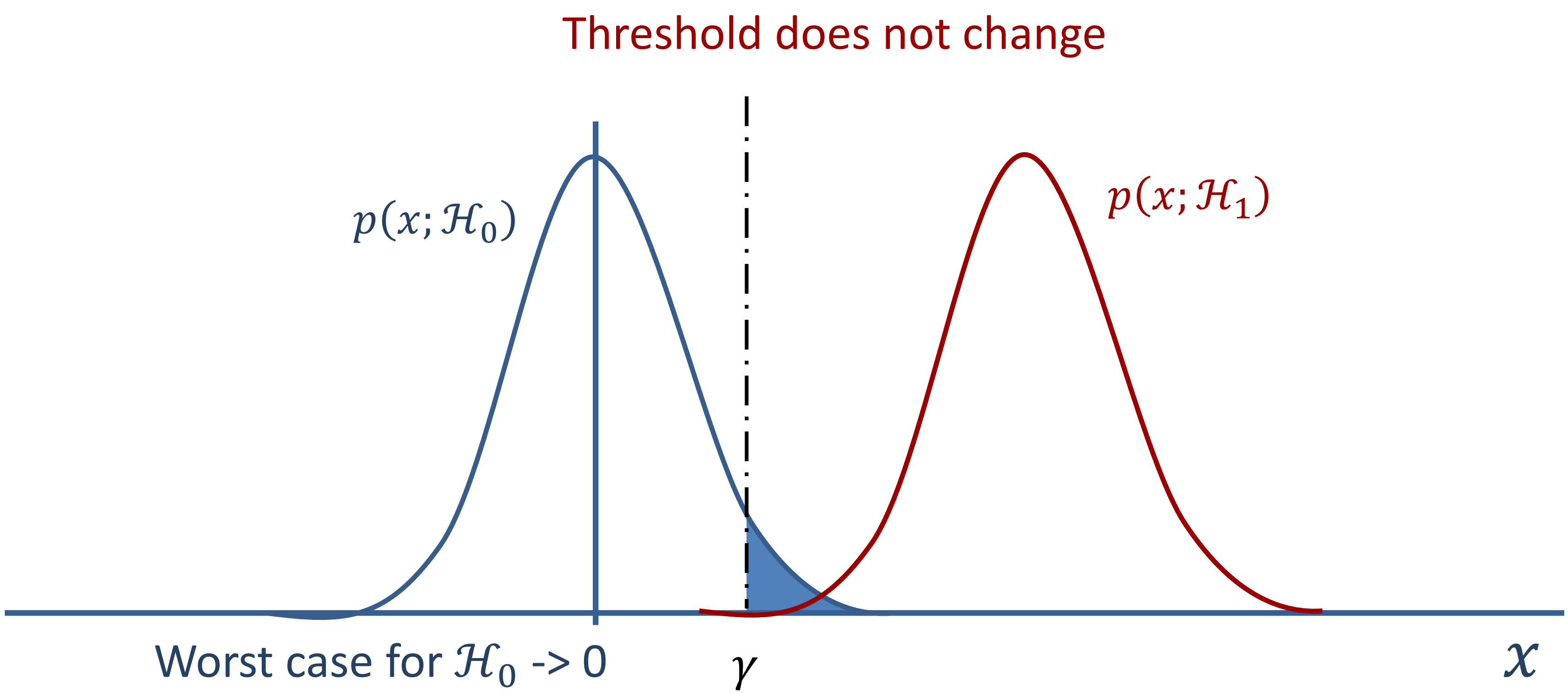
Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$



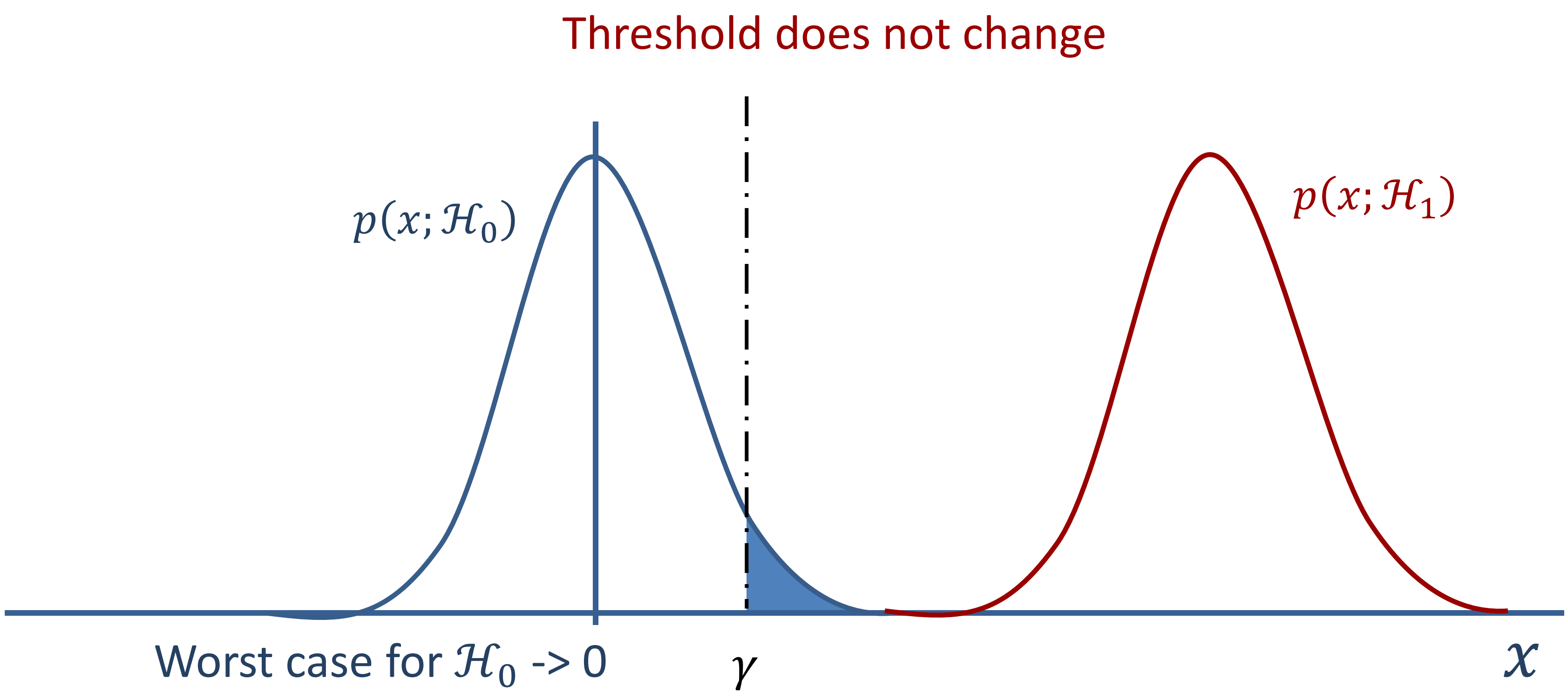
Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$



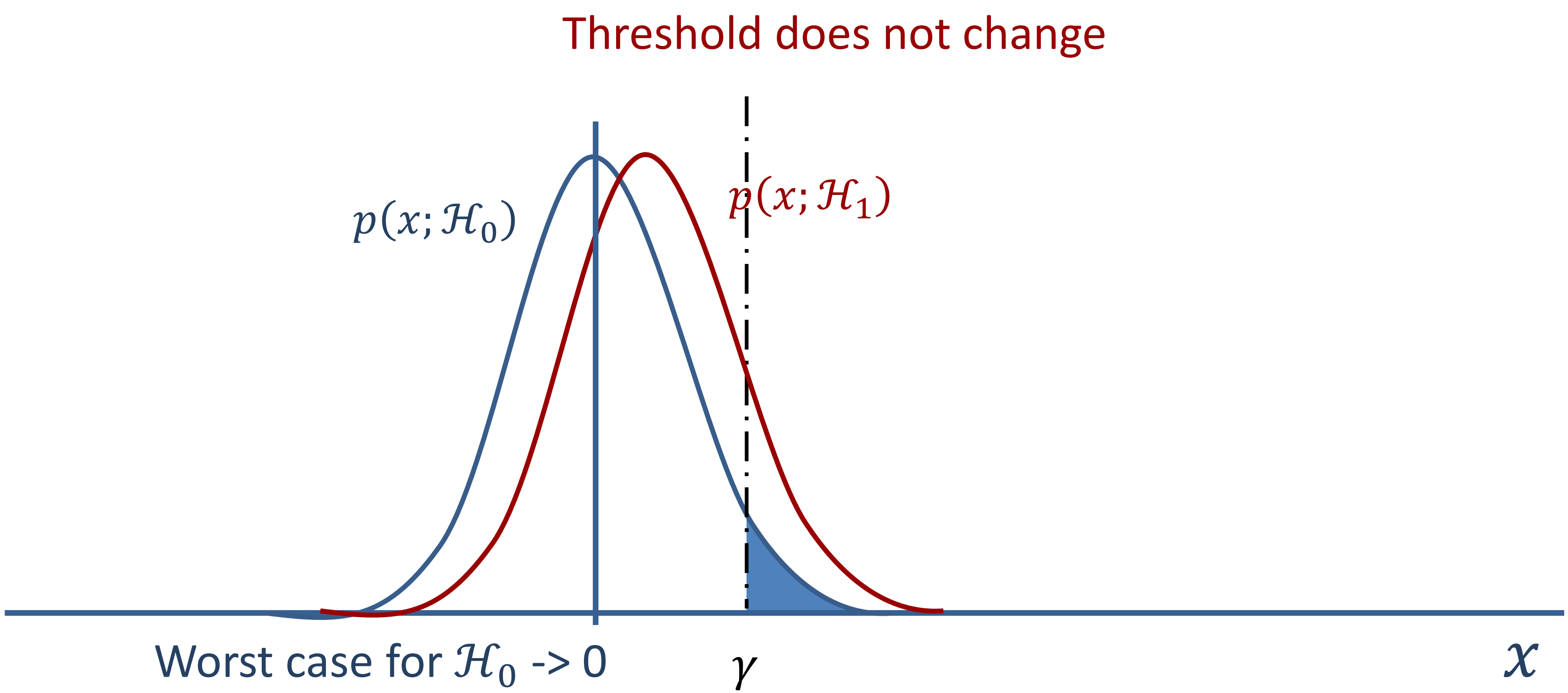
Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$



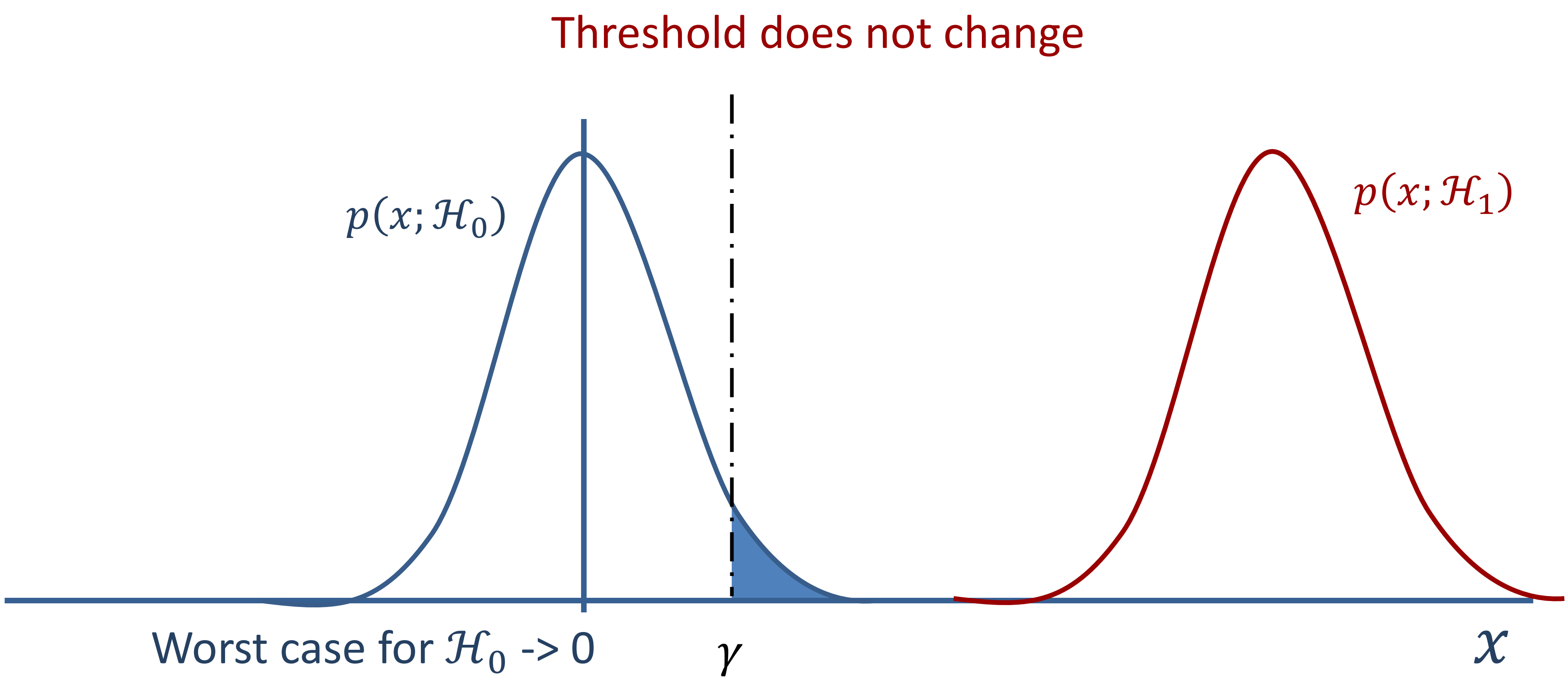
Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$



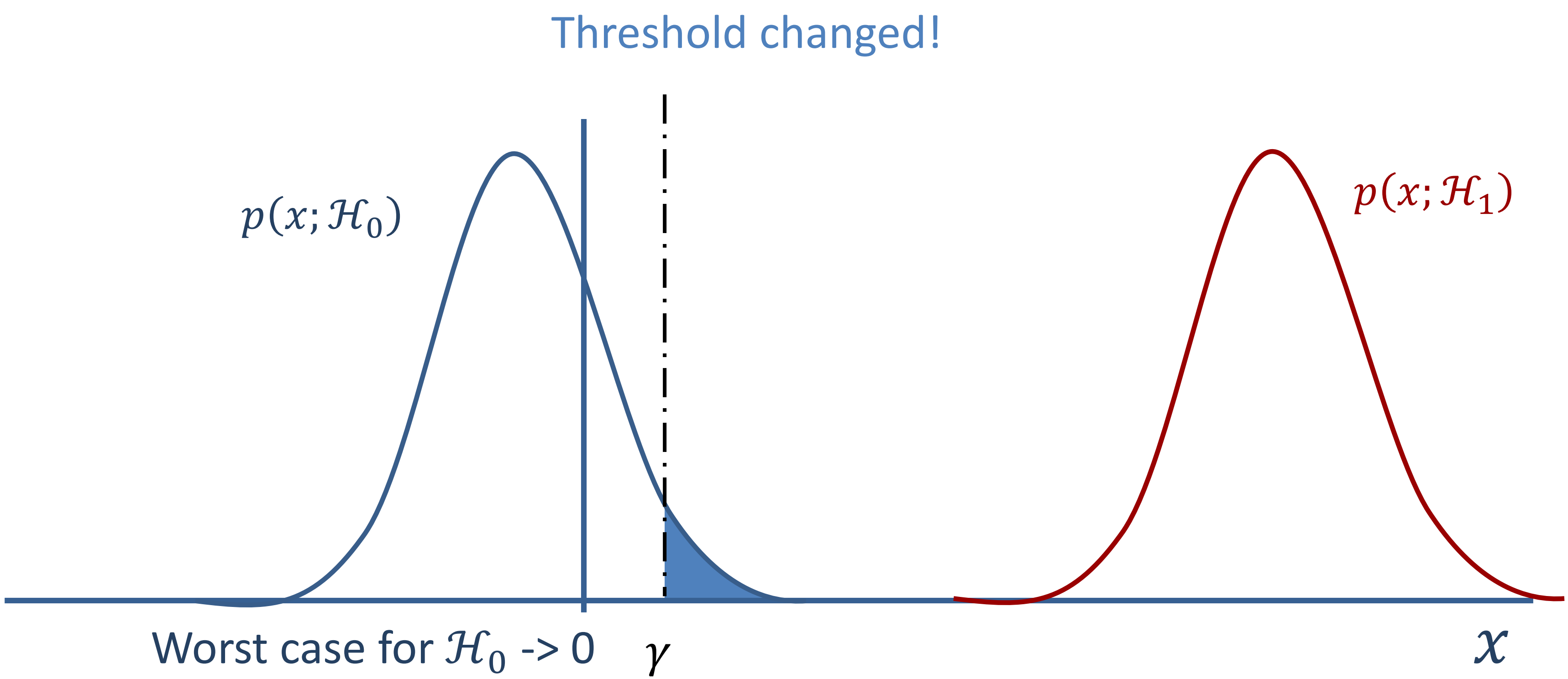
Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$



Composite Hypotheses

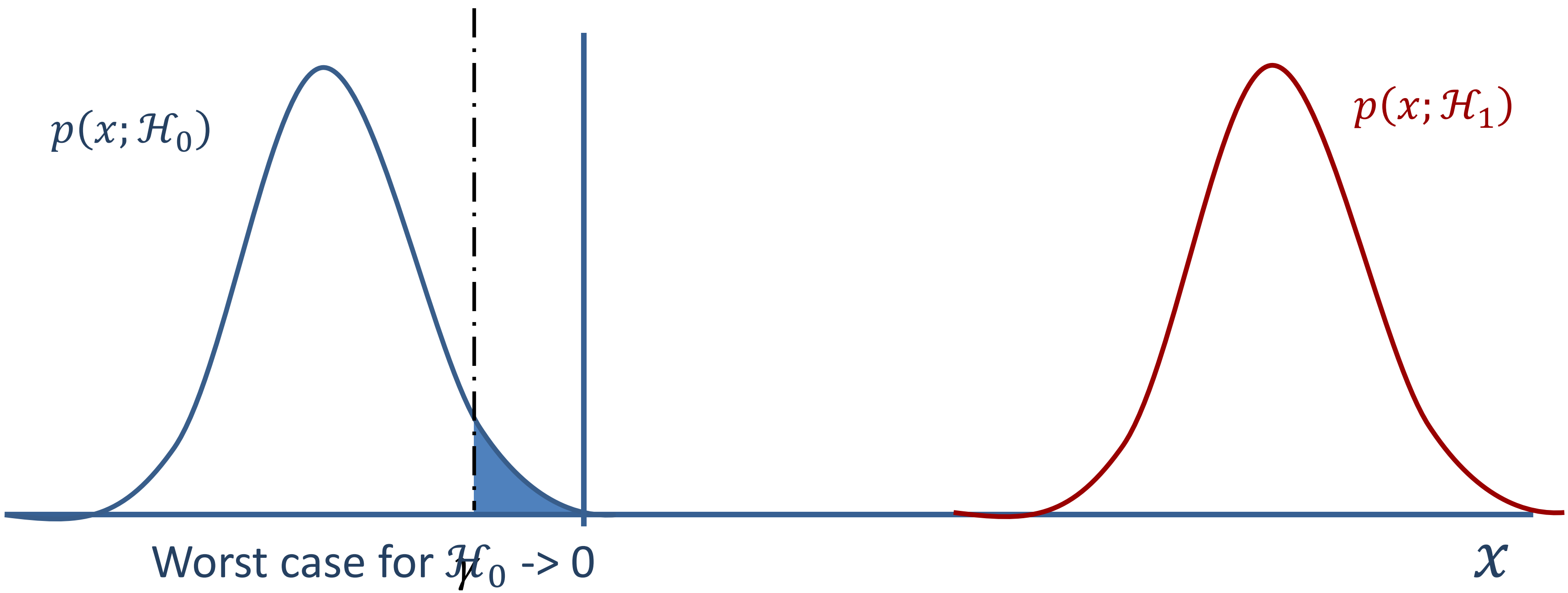
- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$



Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - **Requirement 1:** One-sided problem
  - **Requirement 2:** Data x and parameter θ are scalars
  - **Requirement 3:** The likelihood ratio $p(x;\theta_a)/p(x;\theta_b)$ is monotonically non-decreasing as a function of x for any $\theta_a > \theta_b$

Threshold changed!



Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - Alternatively, we can define the uniformly most powerful (UMP) test as:
    - Let $t = T(\boldsymbol{x})$ be a sufficient statistic for the parameter under test θ

$$\phi(t) = \begin{cases} 1 & \text{if} \quad t > \gamma \\ 0 & \text{if} \quad t < \gamma \end{cases}$$

$$\alpha = E[\phi(t); \theta_0, \mathcal{H}_0]$$

Composite Hypotheses

- **The Karlin-Rubin Theorem**
  - Also allows us create uniformly most powerful invariant statistics tests that remove nuisance parameters $\boldsymbol{\theta}_n$
    - Determine the sufficient statistics for θ and $\boldsymbol{\theta}_n$
    - Determine a function of sufficient statistics $\mathrm{g}\big(T(\boldsymbol{x})\big)$ that does not change with $\boldsymbol{\theta}_n$ (i.e., is invariant to $\boldsymbol{\theta}_n$)
  - Let $t = T(\boldsymbol{x})$ be a sufficient statistic for the parameter under test θ

$$\phi(t) = \begin{cases} 1 & \text{if} \quad t > \gamma \\ 0 & \text{if} \quad t < \gamma \end{cases}$$

$$\alpha = E[\phi(t); \theta_0, \mathcal{H}_0]$$

Example: Matched Filter Again

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$
- **What is the uniformly most powerful (UMP) test?**

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **What is the uniformly most powerful (UMP) test?**

$$p(\boldsymbol{x}) = \left[\frac{1}{(2\pi\sigma^2)^{N/2}}\right] \exp\left[-\frac{1}{2\sigma^2}\|\boldsymbol{x} - \theta \boldsymbol{s}\|^2\right]$$

$$p(\boldsymbol{x}) = \left[\frac{1}{(2\pi\sigma^2)^{N/2}}\right] \exp\left[-\frac{1}{2\sigma^2}(\|\boldsymbol{x}\|^2 - 2\theta \boldsymbol{x}^T \boldsymbol{s} + \theta^2 \|\boldsymbol{s}\|^2)\right]$$

$$p(\boldsymbol{x}) = \underbrace{\left[\frac{1}{(2\pi\sigma^2)^{N/2}}\right] \exp\left[-\frac{1}{2\sigma^2}\|\boldsymbol{x}\|^2\right]}_{h(\boldsymbol{x})} \underbrace{\exp\left[\frac{1}{\sigma^2}\theta \boldsymbol{x}^T \boldsymbol{s} - \theta^2 \|\boldsymbol{s}\|^2\right]}_{g(T(\boldsymbol{x});\theta)}$$

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\quad\quad\quad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **What is the uniformly most powerful (UMP) test?**

$T(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{s}$

So, the UMP test is ??

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{s}$$

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{x}^T\boldsymbol{s} > \gamma \\ 0 & \text{if} \quad \boldsymbol{x}^T\boldsymbol{s} < \gamma \end{cases}$$

$$\alpha = P(\boldsymbol{x}^T\boldsymbol{s} > \gamma; \mathcal{H}_0)$$

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{s}$$

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{x}^T \boldsymbol{s} > \gamma \\ 0 & \text{if} \quad \boldsymbol{x}^T \boldsymbol{s} < \gamma \end{cases}$$

$$\alpha = P(\boldsymbol{x}^T \boldsymbol{s} > \gamma; \mathcal{H}_0) = P\left(\frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\|\sigma} > \frac{\gamma}{\|\boldsymbol{s}\|\sigma}; \mathcal{H}_0\right) = Q\left(\frac{\gamma}{\|\boldsymbol{s}\|\sigma}\right) = Q(\gamma')$$

$$\gamma' = Q^{-1}(\alpha)$$

$$\gamma = Q^{-1}(\alpha)\|\boldsymbol{s}\|\sigma$$

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\quad\quad\quad\quad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **What is the uniformly most powerful (UMP) test?**

The matched filter!

$$T(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{s}$$
$$\gamma = Q^{-1}(\alpha) ||\boldsymbol{s}|| \sigma$$

Or

$$T(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{||\boldsymbol{s}|| \sigma}$$
$$\gamma' = Q^{-1}(\alpha)$$

Example: CFAR (Constant False Alarm Rate) Matched Filter

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$p(\boldsymbol{x}) = \left[\frac{1}{(2\pi\sigma^2)^{N/2}}\right] \exp\left[-\frac{1}{2\sigma^2} \|\boldsymbol{x} - \theta \boldsymbol{s}\|^2\right]$$

$$p(\boldsymbol{x}) = \underbrace{\left[\frac{1}{(2\pi\sigma^2)^{N/2}}\right]}_{h(\boldsymbol{x})} \underbrace{\exp\left[-\frac{1}{2\sigma^2} (\|\boldsymbol{x}\|^2 - 2\theta \boldsymbol{x}^T \boldsymbol{s} + \theta^2 \|\boldsymbol{s}\|^2)\right]}_{g(T(\boldsymbol{x});\theta)}$$

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$
- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T\boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T\boldsymbol{s}}{\|\boldsymbol{s}\|\|\boldsymbol{x}\|/\sqrt{N}} = \frac{\boldsymbol{x}^T\boldsymbol{s}/\|\boldsymbol{s}\|}{\|\boldsymbol{x}\|} \qquad ??$$

How is this distributed?

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

Just multiplying by 1, so it is not affecting our equation

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\| \|\boldsymbol{x}\| / \sqrt{N}} = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\|\boldsymbol{x}/\sigma\| / \sqrt{N}}$$

How is this distributed?

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

Top: $\mathcal{N}(0,1)$

Normal Distribution

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\| \|\boldsymbol{x}\| / \sqrt{N}} = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\|\boldsymbol{x}/\sigma\| / \sqrt{N}}$$

How is this distributed?

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\| \|\boldsymbol{x}\| / \sqrt{N}} = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\|\boldsymbol{x}/\sigma\| / \sqrt{N}}$$

Top: $\mathcal{N}(0,1)$

Normal Distribution

Bottom: $\|\boldsymbol{x}/\sigma\|^2 \sim \mathcal{X}_N^2$

Chi-Squared Distribution

[with N degrees of freedom]

How is this distributed?

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\| \|\boldsymbol{x}\| / \sqrt{N}} = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\|\boldsymbol{x}/\sigma\| / \sqrt{N}} \sim T_N$$

Student's T Distribution
[with N degrees of freedom]

How is this distributed?

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$
- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\| \|\boldsymbol{x}\| / \sqrt{N}} = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\|\boldsymbol{x}/\sigma\| / \sqrt{N}} \sim T_N$$

Student's T Distribution
[with N degrees of freedom]

Statistic not dependent on σ^2 under $\mathcal{H}_0$ (CFAR)!

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad$ $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\| \|\boldsymbol{x}\| / \sqrt{N}} = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\|\boldsymbol{x} / \sigma\| / \sqrt{N}}$$

However, this is dependent on under $\mathcal{H}_1$! So we need to fix this.

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\sqrt{\boldsymbol{x}^T (\boldsymbol{I} - \boldsymbol{P}_s) \boldsymbol{x}} / (\sigma \sqrt{N-1})}$$

However, this is dependent on under $\mathcal{H}_1$! So we need to fix this.

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\underbrace{\sqrt{\boldsymbol{x}^T (\boldsymbol{I} - \boldsymbol{P}_s) \boldsymbol{x}}}_{\boldsymbol{x}^T \boldsymbol{x} - \frac{\boldsymbol{x}^T \boldsymbol{s} \boldsymbol{s}^T \boldsymbol{x}}{\boldsymbol{s}^T \boldsymbol{s}} = \boldsymbol{x}^T \boldsymbol{x} - \frac{|\boldsymbol{x}^T \boldsymbol{s}|^2}{\boldsymbol{s}^T \boldsymbol{s}} = \left\| x - \frac{\boldsymbol{x}^T \boldsymbol{s}}{\boldsymbol{s}^T \boldsymbol{s}} \boldsymbol{s} \right\|^2} / (\sigma \sqrt{N-1})}$$

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w} \qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s}/(\|\boldsymbol{s}\|\sigma)}{\sqrt{\underbrace{\boldsymbol{x}^T(\boldsymbol{I} - \boldsymbol{P}_s)\boldsymbol{x}}}/(\sigma\sqrt{N-1})}$$

$$\boldsymbol{x}^T \boldsymbol{x} - \frac{\boldsymbol{x}^T \boldsymbol{s}\boldsymbol{s}^T \boldsymbol{x}}{\boldsymbol{s}^T \boldsymbol{s}} = \boldsymbol{x}^T \boldsymbol{x} - \frac{|\boldsymbol{x}^T \boldsymbol{s}|^2}{\boldsymbol{s}^T \boldsymbol{s}} = \overbrace{\left\| \boldsymbol{x} - \overbrace{\frac{\boldsymbol{x}^T \boldsymbol{s}}{\boldsymbol{s}^T \boldsymbol{s}}}^{\text{Estimate of } \theta} \boldsymbol{s} \right\|^2}^{\text{Estimate of variance times (N-1)}}$$

Composite Hypotheses

- **Something to keep in mind**

$$\hat{\theta} = \frac{\boldsymbol{x}^T \boldsymbol{s}}{\|\boldsymbol{s}\|^2} = \frac{\boldsymbol{x}^T \boldsymbol{P}_S \boldsymbol{s}}{\|\boldsymbol{s}\|^2}$$

$$\hat{\sigma}^2 = \frac{\boldsymbol{x}^T (I - \boldsymbol{P}_S)\boldsymbol{x}}{N-1}$$

Composite Hypotheses

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{s} + \boldsymbol{w}$ $\qquad\qquad \boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta > 0$

- **Assume σ^2 is unknown. What is the uniformly most powerful (UMP) test?**

$$T(\boldsymbol{x}) = [\boldsymbol{x}^T \boldsymbol{s} \quad \|\boldsymbol{x}\|^2]$$

Can we find a scalar statistic that is invariant to σ^2?

$$T'(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{s} / (\|\boldsymbol{s}\| \sigma)}{\sqrt{\boldsymbol{x}^T (\boldsymbol{I} - \boldsymbol{P}_s) \boldsymbol{x}} / (\sigma \sqrt{N-1})} \sim T_{N-1}$$

Student's T Distribution
[with N-1 degrees of freedom]

Composite Hypotheses

- **Utilizing the Karlin-Rubin Theorem**
  - Allows us to find uniformly most powerful (UMP) tests
  - Allows us to find invariant tests
- **Disadvantages**
  - Only useable for one-sided tests
  - Invariant statistics may not exist or may not be obvious

The Generalized Likelihood Ratio Test

Generalized Likelihood Ratio Tests

- **The Generalized Likelihood Ratio Test**
  - A sub-optimal test
  - Works well in many circumstances
  - Can be used for two-sided tests
  - Inherits many good properties from maximum likelihood estimation

Generalized Likelihood Ratio Tests

- **The generalized likelihood ratio test**
  - Choose $\mathcal{H}_1$ if

$$\frac{\max_{\boldsymbol{\theta}_0} p(\boldsymbol{x}; \boldsymbol{\theta}_0, \mathcal{H}_1)}{\max_{\boldsymbol{\theta}_1} p(\boldsymbol{x}; \boldsymbol{\theta}_1, \mathcal{H}_1)} > \gamma$$

Or

$$\frac{p\left(\boldsymbol{x}; \widehat{\boldsymbol{\theta}}_0, \mathcal{H}_1\right)}{p\left(\boldsymbol{x}; \widehat{\boldsymbol{\theta}}_1, \mathcal{H}_0\right)} > \gamma$$

  - Where $\widehat{\boldsymbol{\theta}}_0$ and $\widehat{\boldsymbol{\theta}}_1$ are the maximum likelihood estimates of $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$

- **Note:**
  - This is generally not an optimal test, but works well in many circumstances

The Generalized Likelihood Ratio Test Example:
Value with unknown weight

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $x = \theta 1 + w \qquad\qquad w \sim \mathcal{N}(0, \sigma^2)$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $x = \theta 1 + w \qquad\qquad w \sim \mathcal{N}(0, \sigma^2)$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} > \gamma$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $x = \theta 1 + w \qquad\qquad w \sim \mathcal{N}(0, \sigma^2)$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max\limits_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} > \gamma$$

  - What is the MLE of θ_1?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $x = \theta 1 + w \qquad\qquad w \sim \mathcal{N}(0, \sigma^2)$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} > \gamma$$

  - What is the MLE of θ_1?
    - $\hat{\theta}_1 = x$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $x = \theta 1 + w \qquad\qquad w \sim \mathcal{N}(0, \sigma^2)$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} = \exp\left[-\frac{1}{2\sigma^2}\left(x - \hat{\theta}_1\right)^2 + \frac{1}{2\sigma^2} x^2\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(x - \hat{\theta}_1\right)^2 + \frac{1}{2\sigma^2} x^2\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(-2x\hat{\theta}_1 + \hat{\theta}_1^2\right)\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(-2x^2 + \hat{\theta}_1^2\right)\right] > \gamma$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $x = \theta 1 + w \qquad\qquad w \sim \mathcal{N}(0, \sigma^2)$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$
- **What is the generalized likelihood ratio test (GLRT)?**

$$x^2 > \gamma'$$

The Generalized Likelihood Ratio Test Example:

Signal with unknown amplitude

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w}$ $\quad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w} \qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} > \gamma$$

What is the MLE of θ_1?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} > \gamma$$

  - What is the MLE of θ_1?
    - $\hat{\theta}_1 = \frac{1}{N}\boldsymbol{x}^T\mathbf{1}$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w} \qquad\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1} p(\boldsymbol{x}; \theta_1, \mathcal{H}_1)}{p(\boldsymbol{x}; \mathcal{H}_0)} = \exp\left[-\frac{1}{2\sigma^2}\left\|\boldsymbol{x} - \hat{\theta}_1 \mathbf{1}\right\|^2 + \frac{1}{2\sigma^2}\|\boldsymbol{x}\|^2\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left\|\boldsymbol{x} - \hat{\theta}_1 \mathbf{1}\right\|^2 + \frac{1}{2\sigma^2}\|\boldsymbol{x}\|^2\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(-2\boldsymbol{x}^T \mathbf{1}\hat{\theta}_1 + N\hat{\theta}_1^2\right)\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(-2\frac{1}{N}|\boldsymbol{x}^T \mathbf{1}|^2 + N\hat{\theta}_1^2\right)\right] > \gamma$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{N}|\boldsymbol{x}^T\mathbf{1}|^2 > \gamma'$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$
- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{N}|\boldsymbol{x}^T\mathbf{1}|^2 > \gamma'$$

What is the distribution of this?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$
- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{N}|\boldsymbol{x}^T\mathbf{1}|^2 > \gamma'$$

What is the distribution of this?

$$\boldsymbol{x}^T\mathbf{1} \sim \mathcal{N}(0, N\sigma^2)$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{N}|\boldsymbol{x}^T\mathbf{1}|^2 > \gamma'$$

What is the distribution of this?

$$\boldsymbol{x}^T\mathbf{1} \sim \mathcal{N}(0, N\sigma^2)$$

$$\frac{1}{N\sigma^2}|\boldsymbol{x}^T\mathbf{1}|^2 \sim \mathcal{X}_1^2$$

Chi-squared with one degree of freedom

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{N}|\boldsymbol{x}^T\mathbf{1}|^2 > \gamma'$$

Question: What is this related to?

The Generalized Likelihood Ratio Test Example:
Signal with unknown amplitude and unknown variance

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

What is the MLE of θ_1?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

What is the MLE of θ_1?

$$\hat{\theta}_1 = \frac{1}{N} \boldsymbol{x}^T \mathbf{1}$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

What is the MLE of σ^2 under $\mathcal{H}_0$?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \boldsymbol{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max\limits_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max\limits_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

What is the MLE of σ^2 under $\mathcal{H}_0$?

$$\hat{\sigma}^2 = \frac{1}{N} \boldsymbol{x}^T \boldsymbol{x}$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

What is the MLE of σ^2 under $\mathcal{H}_1$?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

What is the MLE of σ^2 under $\mathcal{H}_1$?

$$\hat{\sigma}^2 = \frac{1}{N}\left\| \boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1} \right\|^2$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\quad\quad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max\limits_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max\limits_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} > \gamma$$

Okay, let's put this all together now.

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} = \frac{(2\pi\hat{\sigma}_0^2)^{N/2}}{(2\pi\hat{\sigma}_1^2)^{N/2}} \exp\left[-\frac{1}{2\hat{\sigma}_1^2}\left\|\boldsymbol{x} - \hat{\theta}_1\mathbf{1}\right\|^2 + \frac{1}{2\hat{\sigma}_0^2}\|\boldsymbol{x}\|^2\right]$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w} \qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)}$$

$$= \frac{(2\pi\hat{\sigma}_0^2)^{N/2}}{(2\pi\hat{\sigma}_1^2)^{N/2}} \exp\left[-\frac{1}{2\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2 + \frac{1}{2\frac{1}{N}\|\boldsymbol{x}\|^2}\|\boldsymbol{x}\|^2\right]$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w} \qquad\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)}$$

$$= \frac{(2\pi\hat{\sigma}_0^2)^{N/2}}{(2\pi\hat{\sigma}_1^2)^{N/2}} \exp\left[-\cancelto{0}{\frac{1}{2\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2} + \cancelto{0}{\frac{1}{2\frac{1}{N}\|\boldsymbol{x}\|^2}\|\boldsymbol{x}\|^2}\right]$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w} \qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1,\sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} = \frac{(\hat{\sigma}_0^2)^{N/2}}{(\hat{\sigma}_1^2)^{N/2}}$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w} \qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1,\sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} = \frac{(\hat{\sigma}_0^2)^{N/2}}{(\hat{\sigma}_1^2)^{N/2}} = \left(\frac{\frac{1}{N}\boldsymbol{x}^T\boldsymbol{x}}{\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2} \right)^{N/2}$$

  - Note that:

$$\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\right\|^2 = \frac{1}{N}\boldsymbol{x}^T\boldsymbol{x} - 2\frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2 + \frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2 = \boldsymbol{x}^T\boldsymbol{x} - \frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w} \qquad\qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max\limits_{\theta_1,\sigma_1^2} p(\boldsymbol{x};\theta_1,\sigma_1^2,\mathcal{H}_1)}{\max\limits_{\sigma_0^2} p(\boldsymbol{x};\sigma_0^2,\mathcal{H}_0)} = \frac{(\hat{\sigma}_0^2)^{N/2}}{(\hat{\sigma}_1^2)^{N/2}} = \left(\frac{\frac{1}{N}\boldsymbol{x}^T\boldsymbol{x}}{\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2}\right)^{N/2}$$

  - Note that:

$$\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\right\|^2 = \frac{1}{N}\boldsymbol{x}^T\boldsymbol{x} - 2\frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2 + \frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2 = \frac{1}{N}\boldsymbol{x}^T\boldsymbol{x} - \frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2$$

$$\hat{\sigma}_1^2 = \hat{\sigma}_0^2 - \hat{\theta}^2 \qquad \hat{\sigma}_0^2 = \hat{\sigma}_1^2 + \hat{\theta}^2$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w} \qquad\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\max_{\theta_1, \sigma_1^2} p(\boldsymbol{x}; \theta_1, \sigma_1^2, \mathcal{H}_1)}{\max_{\sigma_0^2} p(\boldsymbol{x}; \sigma_0^2, \mathcal{H}_0)} = \frac{(\hat{\sigma}_0^2)^{N/2}}{(\hat{\sigma}_1^2)^{N/2}} = \left(\frac{\hat{\sigma}_1^2 + \hat{\theta}}{\hat{\sigma}_1^2}\right)^{N/2} = \left(1 + \frac{\hat{\theta}^2}{\hat{\sigma}_1^2}\right)^{N/2} > \gamma$$

  - Note that:

$$\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\right\|^2 = \frac{1}{N}\boldsymbol{x}^T\boldsymbol{x} - 2\frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2 + \frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2 = \frac{1}{N}\boldsymbol{x}^T\boldsymbol{x} - \frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2$$

$$\hat{\sigma}_1^2 = \hat{\sigma}_0^2 - \hat{\theta}^2 \qquad \hat{\sigma}_0^2 = \hat{\sigma}_1^2 + \hat{\theta}^2$$

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\quad\quad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\hat{\theta}^2}{\hat{\sigma}_1^2} = \frac{\frac{1}{N^2}|\boldsymbol{x}^T\mathbf{1}|^2}{\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2} = \frac{\left|\frac{1}{N}\boldsymbol{x}^T\mathbf{1}\right|^2}{\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|} = \frac{\frac{1}{N}\left|\frac{1}{\sqrt{N}}\boldsymbol{x}^T\mathbf{1}\right|^2}{\frac{1}{N}\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|} > \gamma'$$

  - What is the distribution of this test statistic under $\mathcal{H}_0$?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w}$ $\qquad\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{\hat{\theta}^2}{\hat{\sigma}_1^2} = \frac{\left|\frac{1}{\sqrt{N}}\boldsymbol{x}^T\mathbf{1}\right|^2}{\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|} > \gamma'$$

  - What is the distribution of this test statistic under $\mathcal{H}_0$?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta\mathbf{1} + \boldsymbol{w} \qquad w \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{\sigma\sqrt{N}}\boldsymbol{x}^T\mathbf{1} \sim \mathcal{N}(0,1) \qquad \left|\frac{1}{\sqrt{N}}\boldsymbol{x}^T\mathbf{1}\right|^2 / \sigma^2 \sim \mathcal{X}_1^2(0,1)$$

$$\frac{\hat{\theta}^2}{\hat{\sigma}_1^2} = \frac{\left|\frac{1}{\sqrt{N}}\boldsymbol{x}^T\mathbf{1}\right|^2}{\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|} > \gamma'$$

$$\left\|\boldsymbol{x} - \frac{1}{N}\boldsymbol{x}^T\mathbf{1}\mathbf{1}\right\|^2 / \sigma^2 \sim \mathcal{X}_{N-1}^2(0,1)$$

  - What is the distribution of this test statistic under $\mathcal{H}_0$?

Generalized Likelihood Ratio Tests

- **Consider the hypotheses**
  - $\boldsymbol{x} = \theta \mathbf{1} + \boldsymbol{w}$ $\qquad w \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I})$
  - $\mathcal{H}_0 : \theta = 0$
  - $\mathcal{H}_1 : \theta \neq 0$

- **Both θ and σ^2 are unknown. What is the generalized likelihood ratio test (GLRT)?**

$$\frac{1}{\sigma\sqrt{N}} \boldsymbol{x}^T \mathbf{1} \sim \mathcal{N}(0,1) \qquad \left| \frac{1}{\sqrt{N}} \boldsymbol{x}^T \mathbf{1} \right|^2 / \sigma^2 \sim \mathcal{X}_1^2(0,1)$$

$$\frac{\hat{\theta}^2}{\hat{\sigma}_1^2} = \frac{\left| \frac{1}{\sqrt{N}} \boldsymbol{x}^T \mathbf{1} \right|^2}{\left\| \boldsymbol{x} - \frac{1}{N} \boldsymbol{x}^T \mathbf{1} \mathbf{1} \right\|} > \gamma'$$

$$\left\| \boldsymbol{x} - \frac{1}{N} \boldsymbol{x}^T \mathbf{1} \mathbf{1} \right\|^2 / \sigma^2 \sim \mathcal{X}_{N-1}^2(0,1)$$

  - What is the distribution of this test statistic under $\mathcal{H}_0$?

$$\frac{\hat{\theta}^2 / 1}{\hat{\sigma}_1^2 / (N-1)} \sim F_{1,N-1}$$

F-distribution
[with 1 and N degrees of freedom]