Full Name: Solutions

ECE 6534 (Spring 2015) – Exam   Date: March 12, 2015

Question	# of Points Possible	# of Points Obtained	Grader
# 1	16		
# 2	18		
# 3	17		
# 4	17		
# 5	16		
# 6	16		
Total	100		

For justifications: Justifications need to show you understand the underlying reasons *why* your answer is correct.

Before starting the exam, read and sign the following agreement.
By signing this agreement, I agree to solve the problems of this exam while adhering to the policies and guidelines of the University of Utah and ECE 6534 and without additional external help. The guidelines include, but are not limited to,

- Notes are allowed
- No textbooks may be used
- No calculators or computers may be used
- No collaboration is allowed
- No cheating is allowed

Student   Date

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Question #1: Let $\mathcal{S}$ be defined by the set of all real, Toeplitz $N \times N$ matrices, a subspace of $\mathbb{R}^{N\times N}$ over the field of real numbers $\mathbb{R}$.

(a) *(5 pts)* Is $\mathcal{S}_1 = \{\alpha I | \alpha \in \mathbb{R}\}$, where I is identity, a subspace of $\mathcal{S}$? **Briefly justify why.**

Yes. S_1 is contained is S (identity is toeplitz) and S_1 is a vector space,

- 0 vector exists
- There is closure under addition + multiplication

(b) *(5 pts)* Is the set of all Toeplitz $N \times N$ matrices with strictly positive elements a subspace of $\mathcal{S}$? **Briefly justify why.**

No. This set does not contain the 0 element, hence it is not a vector space.

(c) *(6 pts)* Determine $\dim(\mathcal{S})$. **Justify why.**

$\dim(S) = 2N-1$ since Toeplitz matrices are uniquely defined by $2N-1$ values (that is, one row and one column (minus a diagonal value).

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Question #2: A permutation matrix P is a $N \times N$ matrix in which exactly one element in each row and each column is equal to 1 and all other elements are equal to 0.

(a) *(5 pts)* (True / False) P has a null space of strictly $\mathcal{N}(P) = \mathbf{0}$. **Justify why.**

True. Every column has one value a unique location (compared with all other columns). Hence, the columns are linearly independent and the matrix is full rank.

(b) *(4 pts)* (True / False) P is an orthogonal operator. **Justify why.**

True. $P^H P = I$ since the n^{th} column of P and n^{th} row of P^H will overlap at one point and the n^{th} column of P and m^{th} row of P^H will never overlap for $n \neq m$. ← resulting in 0 inner product

resulting in an inner product of 1

(c) *(4 pts)* (True / False) P is a projection operator. **Justify why.**

False. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = I \neq P \neq P^2$

Example

Not a projection

(d) *(5 pts)* (True / False) $\sqrt{x^H P x}$ is a norm. **Justify why.**

False. This rearranges the elements in one vector, so we are not gaurenteed $\sqrt{x^H P x} \geq 0$.

Note: If P were an orthogonal projection, it would be true.

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Question #3: Let A be a linear, memoryless, and time-invariant system.
Let $B = D_3U_2U_3D_2A$

(a) *(4 pts)* Is B time-invariant? **Justify why.**

Not common factors, so I can switch

No.

$$D_3U_2U_3D_2A = U_2\cancel{D_3}\cancel{U_3}^{I}D_2A$$

$$= U_2D_2A \rightarrow \text{Removes every other sample from a vector}$$

B is diagonal, but not all values are the same. So B is not time-invariant.

(b) *(4 pts)* Is B memoryless? **Justify why.**

Yes. From (a), B is diagonal, and therefore memoryless.

(c) *(4 pts)* Is B causal? **Justify why.**

Yes. Since it is memoryless, it is causal.

(d) *(5 pts)* Is $D_3U_2U_3D_2$ a projection operator? **Justify why.**

Yes! $P^2 = D_3U_2U_3D_2D_3U_2U_3D_2$

$\boxed{P = P^2} \rightarrow = D_3U_2U_3\cancel{D_6}\cancel{U_6}^{I}D_2$

$= D_3U_2U_3D_2 = P$

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Question #4:

(a) *(6 pts)* Show that $I + A^*A$ is invertible for any $A \in \mathbb{R}^{N\times N}$.

From the HW, we know that A^*A has all real and positive (or zero) eigenvalues.

With eigenvalue decomposition:

$$I + A^*A = UIU^{-1} + U|\Lambda|^2U^{-1} = U(I + |\Lambda|^2)U^{-1}$$

Hence $I + A^*A$ is full rank and invertible ← All diagonal values must ≥ 1

(b) *(5 pts)* Show that if B is a λ-tight frame, then $I - \lambda^{-1}B^*B$ is orthogonal to B^*B.

$(B^*B)^* = B^*B$ ← $(B^*B)^*(I - \lambda^{-1}B^*B) = B^*B - \lambda^{-1*}B^*BB^*B$

By definition of a λ-tight frame: $BB^* = \lambda I$

so $= B^*B - \cancel{\lambda^{-1}}\cancel{\lambda}B^*B = B^*B - B^*B = \boxed{0 = (B^*B)^*(I - \lambda^{-1}B^*B)}$

orthogonal

(c) *(6 pts)* Determine the singular values of D_N. **Justify** why.

D_N is a 1-tight frame since

$$D_N D_N^* = D_N U_N = I$$

Diagonalizing I is trivial (the diagonal is I)
So the signature values must contain 1's or zeros (depending on how we define D_N)

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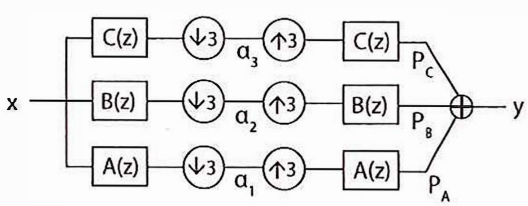
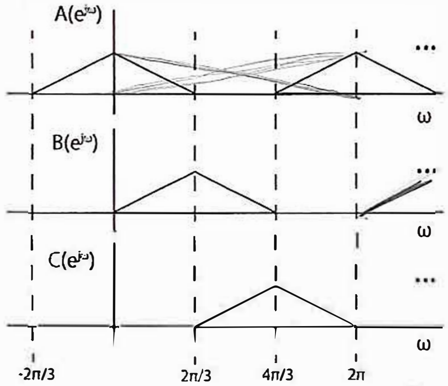
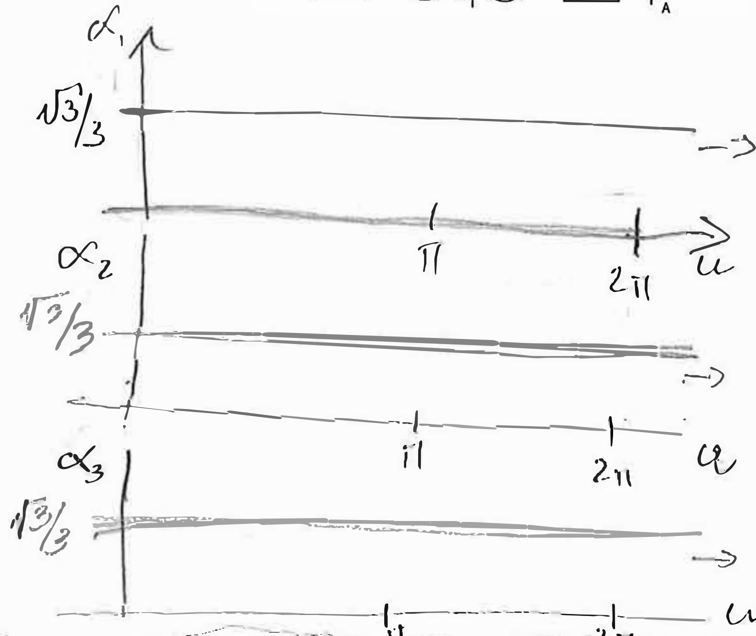
Question #5:

(a) *(8 pts)* Consider the filter bank and filters below. For an impulse input, sketch the frequency response of α_1, α_2, and α_3 as well as P_A, P_B, and P_C. [the maximum amplitude of each frequency response is $\sqrt{3}$]

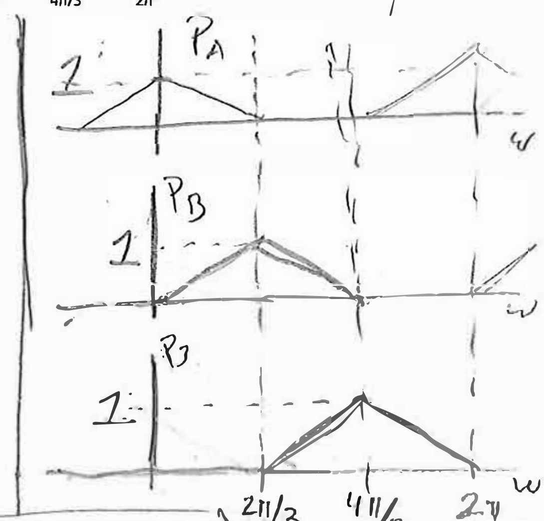




Same as above, but with amplitudes of 1

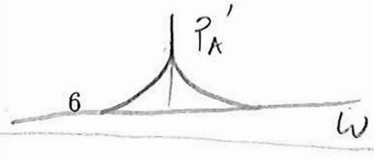


All flat responses!



(b) *(8 pts)* Does the filter bank above satisfy the conditions for perfect reconstruction *in general (for any input x)*? Justify why.

No. There are many ways to show this. For example If I feed P_A back into $\boxed{A(z)} \to (\downarrow 3) \to (\uparrow 3) \to \boxed{A(z)}$, I would not get the same result. It would look something like



$\boxed{\text{Hence, each branch is } \underline{\text{not}} \text{ a projection. No perfect reconstruction.}}$

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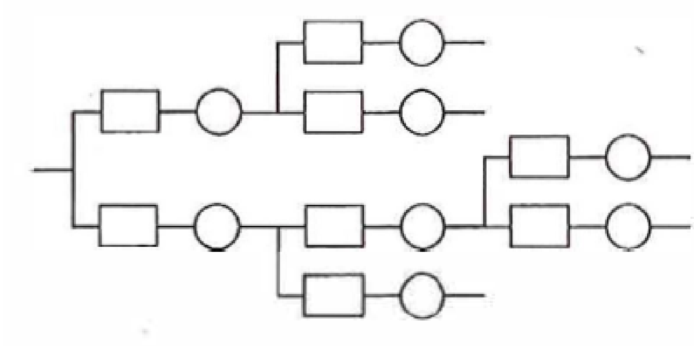
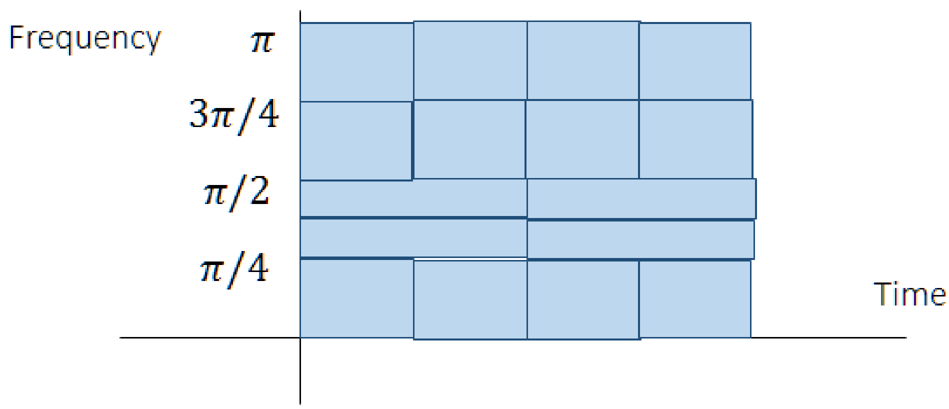
Question #6:

(a) *(8 pts)* Sketch the time-frequency tiling for the following wavelet packet. Assume the top filters are high pass and the bottom filters are low pass and the downsamplings are by 2.





2:41

(b) *(8 pts)* Design a wavelet packet tree for the following time-frequency tiling.