Full Name: Solutions

ECE 6534 (Spring 2017) – Exam **Date: March 20, 2017**

Question	# of Points Possible	# of Points Obtained	Grader
# 1	20		
# 2	19		
# 3	20		
# 4	20		
# 5	21		
Total	100		

For justifications: Justifications need to show you understand the underlying reasons *why* your answer is correct.

Before starting the exam, read and sign the following agreement.
By signing this agreement, I agree to solve the problems of this exam while adhering to the policies and guidelines of the University of Utah and ECE 6534 and without additional external help. The guidelines include, but are not limited to,

- One 8.5 inch by 11 inch cheat sheet is allow
- No textbooks may be used
- No calculators or computers may be used
- No collaboration is allowed
- No cheating is allowed

Student

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Question #1: Let $\mathcal{S}_2$ and $\mathcal{S}_1$ be defined by the following sets

$$\mathcal{S}_1 = \{ \quad x = |\alpha| + 1 \quad | \quad \alpha \in \mathbb{R} \quad \} \quad , \quad \mathcal{S}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

(a) *(5 pts)* Is $\mathcal{S}_1$ a subspace of $\mathbb{R}$? **Briefly justify why.**

I can rewrite this as $S_1 = \{ x \geq 1 \}$

S_1 does not contain the zero value, therefore it is not a vector space and $\boxed{\text{not a subspace}}$,

(b) *(5 pts)* Are the elements of $\mathcal{S}_1$ linearly independent? **Justify why.**

$\boxed{\text{No}}$, the elements are linearly dependent. Every element is a scalar so I can multiply by another scalar to get any other element in S_1.

(c) *(5 pts)* Determine $\dim(\mathcal{S}_2)$. **Justify why.**

$$\boxed{Dim(S_2) = 2}$$

since ~~column~~ vectors 2, 3 are linearly independent

but $-2v_3 + 3v_1 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = v_2$ ← linearly dependence

(d) *(5 pts)* Are the elements of $\mathcal{S}_2$ orthogonal? **Justify why.**

$\boxed{\text{No}}$ $\langle v_1, v_2 \rangle = 0(2) + (1)(-1) + (0)(2)$

$= -1$ ← not zero

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Question #2: Let I be the $N \times N$ identity matrix, where all of the diagonal values (top-left to bottom-right) are 1. Let J be the $N \times N$ exchange matrix, where all of the anti-diagonal values (top-right to bottom-left) are 1. Assume $N = 4$ and let $A = (I + J)/\sqrt{2}$.

(a) *(6 pts)* Determine the range space of A (i.e., $\mathcal{R}(A)$).

$$A = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\mathcal{R}(A) = \left\{ \begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} = \left\{ \text{all vectors with even symmetry} \right\}}$$

(b) *(6 pts)* Determine the null space of A (i.e., $\mathcal{N}(A)$).

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ -d \\ -c \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} c + 0 + 0 - c \\ 0 + d - d + 0 \\ 0 + d - d + 0 \\ c + 0 + 0 - c \end{bmatrix} = \vec{0}$$

$$\boxed{\mathcal{N}(A) = \left\{ \begin{bmatrix} c \\ d \\ -d \\ -c \end{bmatrix} \,\middle|\, c, d \in \mathbb{R} \right\} = \left\{ \text{all vectors with odd symmetry} \right\}}$$

(c) *(7 pts)* (True / False) A is a projection operator. **Justify why.**

$$A^2 = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \sqrt{2} A$$

$A^2 = \sqrt{2} A \neq A$

Not a projection (almost though)

False

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Question #3: Consider a finite-length rank-one matrix defined by

$$R = vv^H$$

where v is an $N \times 1$ vector.

$v = [a\ a\ a\ a]^T$ then $vv^T = \begin{bmatrix} a^2 & a^2 & a^2 & a^2 \\ a^2 & a^2 & a^2 & a^2 \\ a^2 & a^2 & a^2 & a^2 \\ a^2 & a^2 & a^2 & a^2 \end{bmatrix}$

(a) *(7 pts)* Can R be time-invariant? If so, with what condition on the values of v is this true?

Example: $v = [a\ b\ c\ d]^T$

Then

$$vv^H = [a\ b\ c\ d]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}$$

symmetric

This can be time-invariant when all values of v are equal

=> matrix is then circulant

(b) *(7 pts)* Can R be memoryless? If so, with what condition on the values of v is this true?

if $v = [1\ 0\ 0\ 0]^T$

$$vv^H = [1\ 0\ 0\ 0]\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Diagonal matrix

This can be memoryless when all but one value of v is zero

(c) *(6 pts)* Can R be causal? If so, with what condition on the values of v is this true?

This can be causal when the system is memoryless. Otherwise, it cannot be causal due to the symmetry of vv^H.

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Question #4:

(a) *(7 pts)* (True / False) If the eigenvaluess of a matrix H are only 1 and -1, then H is an orthogonal matrix. **Justify why.**

If H is orthogonal, then

need $U^* = U^{-1}$ to get I

$$H^*H = I = (U\Lambda U^{-1})^*(U\Lambda U^{-1}) = U^{-*}\Lambda^* U^* U \Lambda U^{-1}$$

$$= U\Lambda^*\Lambda U^{-1} = U\Lambda^2 U^{-1} = UIU^{-1} = I$$

square of 1 and −1 is 1

$\boxed{\text{So True}}$

(b) *(6 pts)* (True / False) The product of all singular values of a matrix is a norm. **Justify why.**

$\boxed{\text{False}}$

$\prod_{i=1}^{N} \sigma_i$ can equal zero if a single $\sigma_i = 0$. This voilates the $\|x\| = 0$ iff $x = 0$.

(c) *(7 pts)* (True / False) A tight frame must be full rank (i.e., it must have the maximum possible number of linearly independent columns). **Justify why.**

$\boxed{\text{True}}$

$H = U\Sigma V^*$. A tight frame satisfies $HH^* = I$.

So $HH^* = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma \underbrace{V^*V}_{I}\Sigma^* U^*$

$= U\Sigma\Sigma^* U^*$ ← To be $= I$, Σ needs all of its sigular values, i.e., H needs to be full rank

$= UIU^* = I$

Example:

$$\underset{\Sigma}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underset{\Sigma^*}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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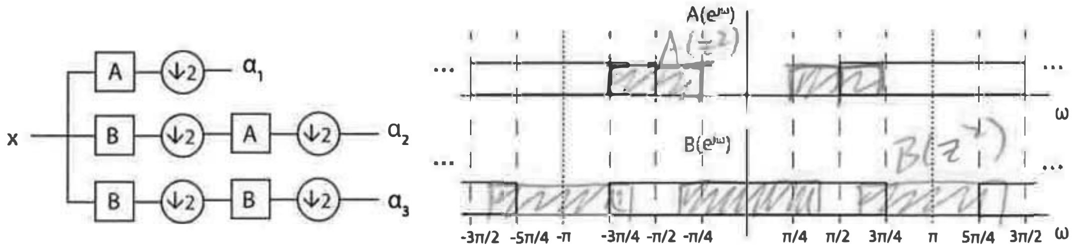
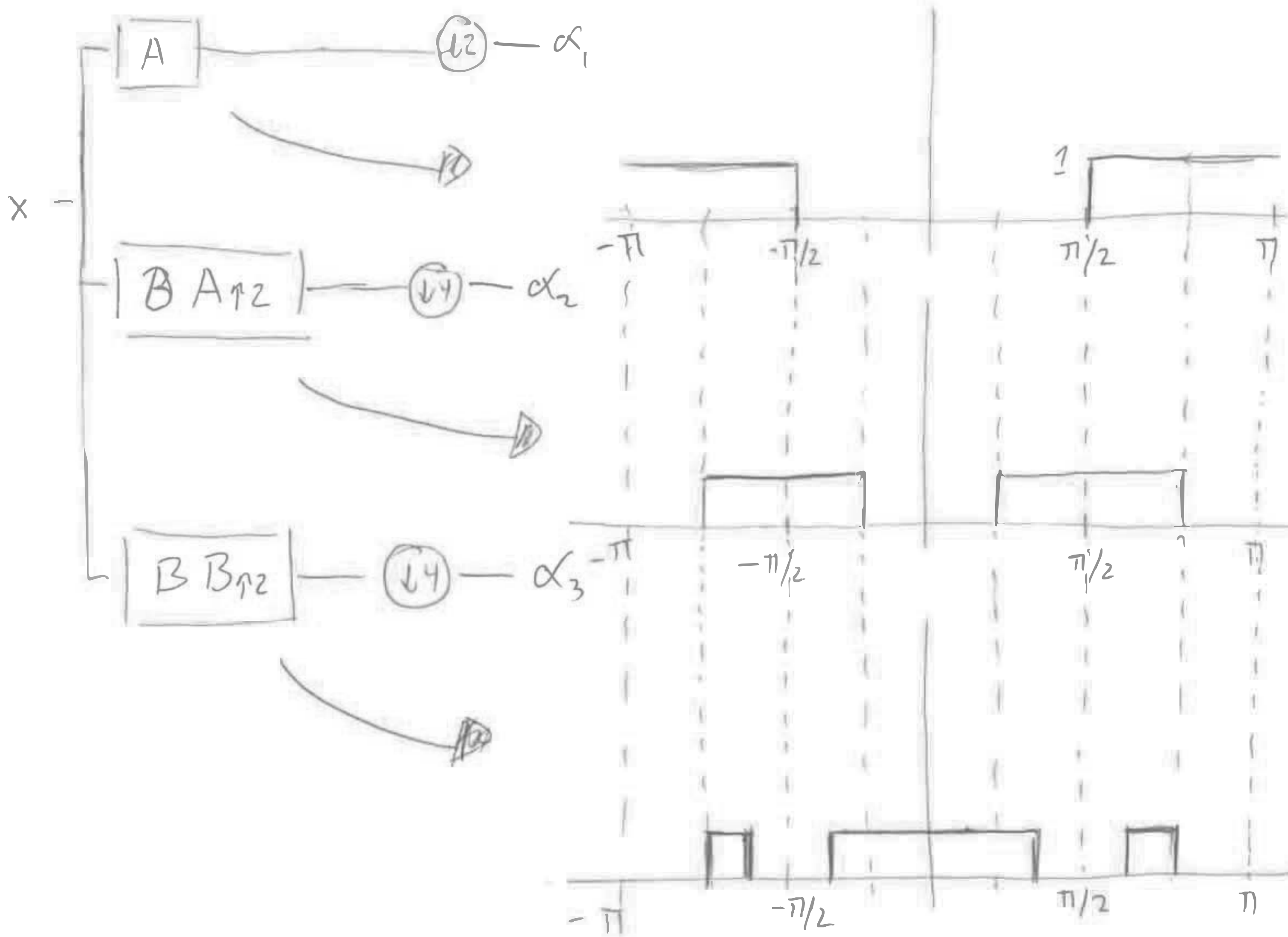
Question #5:

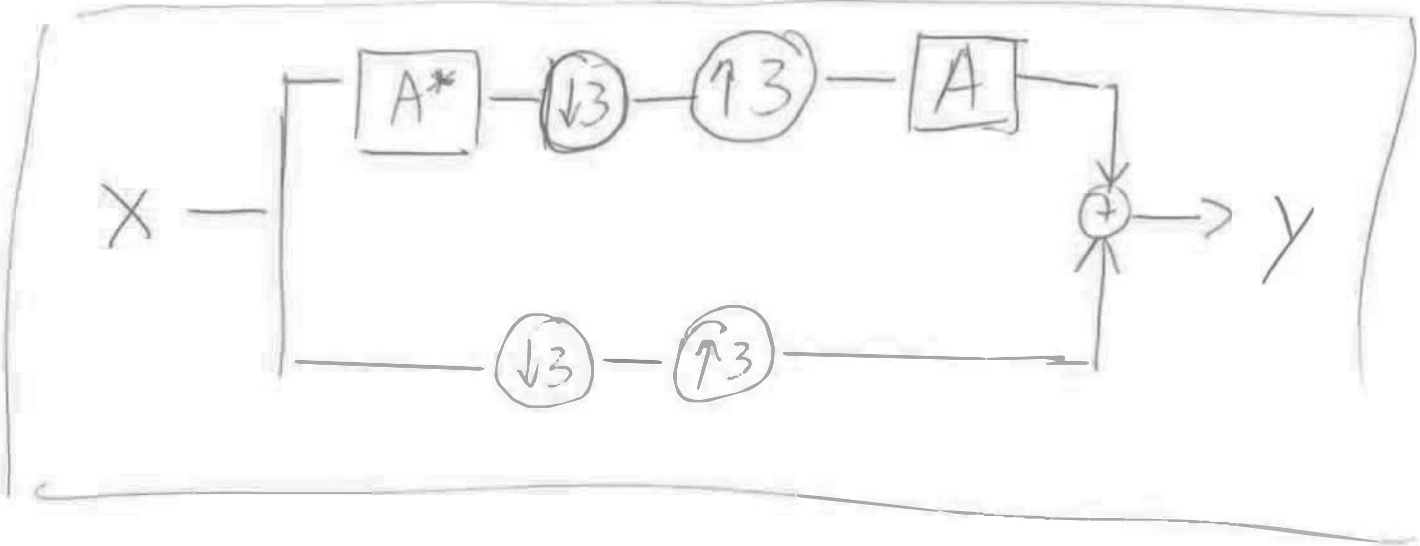
(a) *(10 pts)* Consider the filter bank and filters below. Simplify each branch. Draw the equivalent, simplified filter bank and sketch the equivalent filter frequency responses.



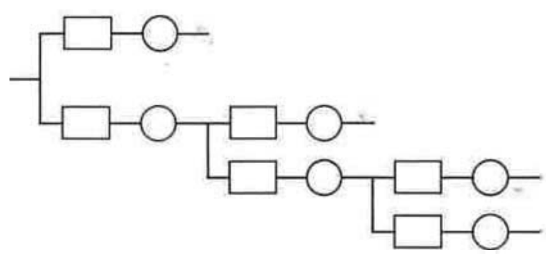


(b) *(5 pts)* Sketch the block diagram for

$$y = AU_3D_3A^*x + U_3D_3x$$



(c) *(6 pts)* Sketch the time-frequency tiling for the following wavelet packet. Assume the top filters are high pass and the bottom filters are low pass and the downsamplings are by 2.



Time- freq. tiling