ECE5340/6340: Homework 4
Numerical Integration

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February 1, 2012

Write your section (ECE5340 or ECE6340) by your name. Turn in a printed copy containing the problem solutions, plots, and the code used to generate them. Remember to comment and format the code so is legible to the graders. Label the plots appropriately, including units for each axis and for the values plotted. Assume all units to be SI units unless stated differently. Due Wednesday 2/8 BEFORE class begins.

ASSIGNMENT

1. Re-derive the error bound for the midpoint rule and show that it satisfies

$$\epsilon \approx -\frac{1}{24} f^{(2)}(c) \Delta x^3 \ .$$

2. For a single trapezoid of width Δx and heights $f(a)$ and $f(b)$, the area is given as simply $A = \Delta x(f(a)+f(b))/2$. From this knowledge, show that if we subdivide the interval $[a, b]$ into n subintervals, then the area under the function $f(x)$ may be approximated as

$$A' = \Delta x \left[\frac{f(a)}{2} + \frac{f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right] \ ,$$

where $x_i = a + i\Delta x$.

3. Derive the error bound for the trapezoidal rule and show that it satisfies

$$\epsilon \approx \frac{1}{12} f^{(2)}(c) \Delta x^3 \ .$$

4. Write a Matlab code that calculates the numerical approximation to a definite integral using the left-point rule. Repeat for the midpoint rule, trapezoidal rule, and Simpson's rule. You will use these in the following sections (Note: Matlab has a built-in functions like "trapz" that implement numerical integration algorithms. You can compare against these functions for debugging purposes, but be sure to write your own code).

5. Consider the definite integral

$$A_k = \int_1^2 f_k(x)\, dx \ .$$

Solve for A_k using the following set of functions:

$$\begin{aligned} f_1(x) &= x \\ f_2(x) &= x^2 \\ f_3(x) &= x^3 \\ f_4(x) &= x^4 \\ f_5(x) &= \sin x \end{aligned}$$

These functions will serve as our test functions for computing numerical integrals. Fill in your analytical solutions using Table 1 at the end of this document.

6. Use your numerical integration codes to calculate the approximate areas for each A_k using the trapezoidal rule and Simpson's rule. Fill in your calculations using Table 1 at the end of this document.

7. Calculate the expected error for both the Trapezoidal rule and Simpson's rule that should arise from using $n = 10$ sub-intervals. For simplicity, only calculate the error you would expect at the midpoint $c = (a + b)/2$ and then multiply by the total number of subintervals n. Compare these errors against the true errors by using your Matlab programs. Fill in your solutions using Table 2.

8. Some important mathematical functions are very difficult to calculate analytically. For example, consider the *error function* defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \ .$$

   Calculate the absolute error $|\epsilon|$ of a numerical approximation to $\text{erf}(x)$ as a function of increasing subintervals using $[a, b] = [0, 1]$. Compare the left-point rule, the trapezoid rule, and Simpson's rule against the exact value (hint: use Matlab's built-in function "erf" to find the true area). Plot your results on a log-log scale from $n = 2$ subintervals up to $n = 1000$.

9. Download the AM 1.0 spectral irradiance data contained in the Matlab file *solarData.mat* from the website. This data represents the instantaneous influx of solar radiation to an observer on Earth with the sun directly overhead. Use numerical integration on the data set to calculate how much energy the sun is delivering to us. Note that the wavelength units are given in nanometers and the power flux is given in units of $\text{W/m}^2\cdot\text{nm}$ (Be careful! The wavelength data is not uniformly spaced).

10. <u>ECE 6340 Only</u>: Analytically solve the integral

$$A_k = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_k(x, y)\, dy dx \ ,$$

   over the domain $[x_1, x_2] = [0, 1]$ and $[y_1, y_2] = [0, 3]$ for the following set of functions:

$$\begin{aligned} f_1(x, y) &= xy \\ f_2(x, y) &= x^4 y^4 \\ f_3(x, y) &= e^{xy} \end{aligned}$$

   NOTE: f_3 is not a separable function, and therefore cannot be solved by hand. So just use a value of $A_3 \approx 8.258$ to verify your code.

11. <u>ECE 6340 Only</u>: Write a Matlab program that calculates the numerical integral of a 2D function by applying the trapezoidal rule. Compare your numerical results against the exact values by filling in Table 3. Use $\Delta x = \Delta y = 0.01$ for setting up your integration.

$f(x)$	Analytical Solution	Trapezoidal Rule	Simpson's Rule
x			
x^2			
x^3			
x^4			
$\sin x$			

Table 1: 1-D numerical integration summary using $n = 10$ on the interval $[a, b] = [1, 2]$.

$f(x)$	Trapezoidal Error (Expected)	Trapezoidal Error (Observed)	Simpson's Error (Expected)	Simpson's Error (Observed)
x				
x^2				
x^3				
x^4				
$\sin x$				

Table 2: Error summary.

$f(x, y)$	Analytical Solution	Trapezoidal Rule
xy		
x^4y^4		
e^{xy}	≈ 8.258	

Table 3: 2-D integration summary.