THE GRADIENT

SUPPOSE THAT A GIVEN VECTOR FUNCTION $\bar{F}(x,y,z)$ HAS AN ASSOCIATED SCALAR FUNCTION $\psi(x,y,z)$ AND THAT THESE RELATE SO

$$F_x = \frac{\partial \psi}{\partial x} \qquad F_y = \frac{\partial \psi}{\partial y} \qquad F_z = \frac{\partial \psi}{\partial z}$$

WHERE $\bar{F}(x,y,z) = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

CONSIDER NOW THE LINE INTEGRAL OF $\bar{F}\cdot\hat{t}$ AS WE DID WHEN TALKING ABOUT CIRCULATION:

$$\int_C \bar{F}\cdot\hat{t}\, dc$$

WHERE C IS THE PATH OF INTEGRATION, C IS THE CURVE PARAMETER, AND $\hat{t}$ IS THE TANGENT VERSOR AT EACH POINT OF THE CURVE

$$\hat{t} = \frac{\partial x}{\partial c}\hat{i} + \frac{\partial y}{\partial c}\hat{j} + \frac{\partial z}{\partial c}\hat{k}$$

THEN, $\bar{F}\cdot\hat{t}$ CAN BE EXPRESSED AS

$$\bar{F}\cdot\hat{t} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial c} = \frac{\partial \psi}{\partial c}$$

CONSIDER INTEGRATING NOW $\bar{F}\cdot\hat{t}$ BETWEEN TWO POINTS OF THE CURVE C, SAY, P_0 AND P_1

$$\int_C \bar{F}\cdot\hat{t}\, dc = \int_C \frac{d\psi}{dc}\, dc = \int_C d\psi = \psi(P_1) - \psi(P_0)$$

SO WE CAN SEE NOW THAT THE VALUE OF THIS INTEGRAL DEPENDS OF THE INITIAL AND FINAL POINTS, BUT NOT FROM THE PATH, AS dc WAS SIMPLIFIED.

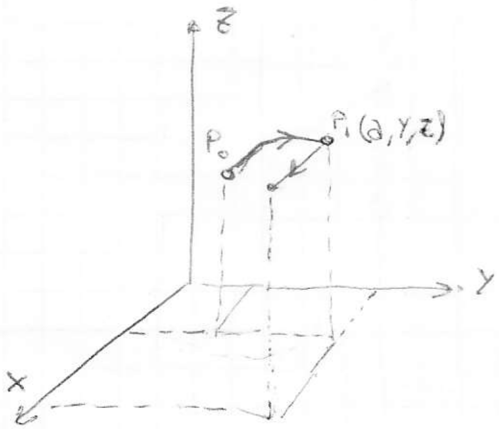
IF WE CONSIDER P_0 AS A REFERENCE POINT, WE CAN ALSO WRITE

$$\psi(x,y,z) = \int_{P_0}^{P_1} \bar{F} \cdot \hat{t}\, dc$$

THIS IS ONLY VALID BECAUSE WE KNOW THE RESULT IS PATH-INDEPENDENT.

IN GENERAL, TO EVALUATE THIS INTEGRAL, WE CAN CHOOSE ANY ARBITRARY PATH OF INTEGRATION, AS WE KNOW THAT OUR CHOICE WILL NOT AFFECT THE RESULT, PROVIDING WE START ALWAYS FROM P_0 AND FINISH IN OUR POINT OF INTEREST.

WE CHOOSE, THEN, A PATH THAT HAS ONE PORTION OF IT THAT REDUCES TO A 1-D INTEGRAL ON THE $\hat{x}$ DIRECTION.



OUR INTEGRAL FROM P_0 TO P CAN THEN BE EVALUATED AS

$$\psi(x,y,z) = \underbrace{\int_{P_0}^{P_1} \bar{F} \cdot \hat{t}\, dc}_{\text{INDEPENDENT OF } x \text{, SO } \frac{\partial}{\partial x} = 0} + \underbrace{\int_{P_1}^{P} F_x(x,y,z)\, dx}_{\text{VARYING ONLY IN } x}$$

BUT

$$\int_{P_1}^{P} F_x(x,y,z)\, dx = \int_{a}^{x} F_x(x,y,z)\, dx$$

↑↑ CONSTANTS

AND IF WE CONSIDER ONLY THE VARIATION IN X,

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \int_a^x F_x(x,y,z)\, dx = F_x(x,y,z) \implies \boxed{F_x = \frac{\partial \psi}{\partial x}}$$

SIMILARLY, CHOOSING THE APPROPRIATE INTEGRATION PATHS,

$$F_y = \frac{\partial \psi}{\partial y} \quad \text{AND} \quad F_z = \frac{\partial \psi}{\partial z}$$

SO, SINCE

$$\bar{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

WE HAVE

$$\bar{F} = \frac{\partial \psi}{\partial x}\hat{i} + \frac{\partial \psi}{\partial y}\hat{j} + \frac{\partial \psi}{\partial z}\hat{k} = \boxed{\underbrace{\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\psi}_{\nabla \psi \quad \text{"DEL PSI"}}}$$

WHICH IS THE GRADIENT OF ψ IN CARTESIAN COORDINATES, WHICH IS A VECTOR FUNCTION $\bar{F}$.

WE CAN SEE THAT THERE IS A RELATION BETWEEN PATH INDEPENDENCE AND THE EXISTENCE OF A SCALAR FUNCTION

$$\psi(x, y, z)$$

SUCH THAT

$$\boxed{\bar{F} = \nabla \psi}$$

IF WE CONSIDER A SIMPLY CONNECTED REGION, IT CAN BE DEMONSTRATED THAT IF

$$\nabla \times \bar{F} = 0$$

THEN THERE EXIST A SCALAR FUNCTION $\psi(x, y, z)$ SUCH THAT

$$\bar{F} = \nabla \psi$$

WE CAN USE THIS KNOWLEDGE TO DETERMINE THE ELECTROSTATIC POTENTIAL ASSOCIATED WITH AN $\bar{E}$ FIELD. SINCE

$$\nabla \times \bar{E} = 0$$

WE CAN SAY THAT $\boxed{\bar{E} = -\nabla V}$

WHERE SINCE ∇V IS A VECTOR IN THE DIRECTION OF INCREASING V, THE FORCE ON A POSITIVE CHARGE q IS

$$\bar{F} = q\bar{E} = -q\nabla V$$

WHICH IS IN THE DIRECTION OF DECREASING V. THE NEGATIVE SIGN ENSURES THAT A POSITIVE CHARGE MOVES "DOWNHILL" FROM A HIGHER TO A LOWER POTENTIAL.

LAPLACIAN, LAPLACE EQUATION, POISSON EQUATION

COMBINING

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \quad \text{AND} \quad \bar{E} = -\nabla V$$

WE OBTAIN

$$\nabla \cdot (\nabla V) = -\frac{\rho}{\varepsilon_0} \quad (1)$$

EXPANDING THE LEFT-HAND SIDE,

$$\nabla \cdot (\nabla V) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right]$$

$$\nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

WE CAN WRITE THIS IN A MORE COMPACT WAY BY INTRODUCING A NEW OPERATOR CALLED '<u>LAPLACIAN</u>', DENOTED AS ∇^2 (READ "DEL SQUARED")

$$\nabla^2 = \nabla \cdot \nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z}$$

THEN, (1) CAN BE WRITTEN AS

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}} \quad \text{POISSON EQUATION.}$$

A GENERAL DEFINITION OF THE LAPLACIAN OPERATOR IS

$$\boxed{\nabla^2 F = \nabla \cdot (\nabla F)}$$ DEFINITION OF LAPLACIAN OPERATOR.

IF WE CONSIDER THE POISSON EQUATION IN A MEDIUM WITH NO CHARGES,

$$\rho = 0 \quad \Rightarrow \quad \boxed{\nabla^2 V = 0}$$ LAPLACE EQUATION

THIS EXPRESSION IS KNOWN AS THE LAPLACE EQUATION.