LINE INTEGRALS AND VECTOR FUNCTIONS

WE CAN USE LINE INTEGRALS TO CALCULATE THE WORK DONE BY A VECTOR FUNCTION OVER A PATH (CURVE).

FROM THE DEFINITION OF WORK,

$$W = F d \qquad [J = Nm]$$

WHERE W IS THE WORK DONE IN JOULES. IF WE CONSIDER A CURVED PATH, WE CAN USE A LINE INTEGRAL. IN THE CASE OF CALCULATING THE WORK DUE TO A VECTOR FIELD,

$$W = \int_C \bar{F}(x,y,z) \cdot \hat{t}(x,y,z)\, dc$$

WHERE W IS THE WORK DONE, C IS THE PATH OF INTEGRATION, $\bar{F}$ IS THE VECTOR FUNCTION, AND $\hat{t}(x,y,z)$ IS THE TANGENT VECTOR TO C AT POINT (x,y,z)

THE TANGENT VECTOR IS DEFINED AS:

$$\hat{t}(c) = \hat{i}\frac{dx}{dc} + \hat{j}\frac{dy}{dc} + \hat{k}\frac{dz}{dc}$$

IF THE LINE PATH IS CLOSED, WE CALL THIS EXPRESSION 'CIRCULATION' OF $\bar{F}$ OVER C.

$$\boxed{\oint_C \bar{F} \cdot \hat{t}\, dc \qquad \text{CIRCULATION OF } \bar{F} \text{ OVER } C}$$

IF THE FIELD IS CONSERVATIVE (THE LINE INTEGRAL BETWEEN TWO POINTS IS PATH INDEPENDENT), THE FOLLOWING IS TRUE:

$$\oint_C \bar{F} \cdot \hat{t}\, dc = 0$$

IN THIS CASE:
1. $\bar{F}$ DEPENDS ON DISTANCES BETWEEN PARTICLES.
2. IT ACTS ALONG THE LINE JOINING THEM.

IN ADDITION, IT CAN BE PROVEN THAT ON CONSERVATIVE FIELDS $\nabla \bar{F} = \varphi$, WHERE φ IS A SCALAR FUNCTION.

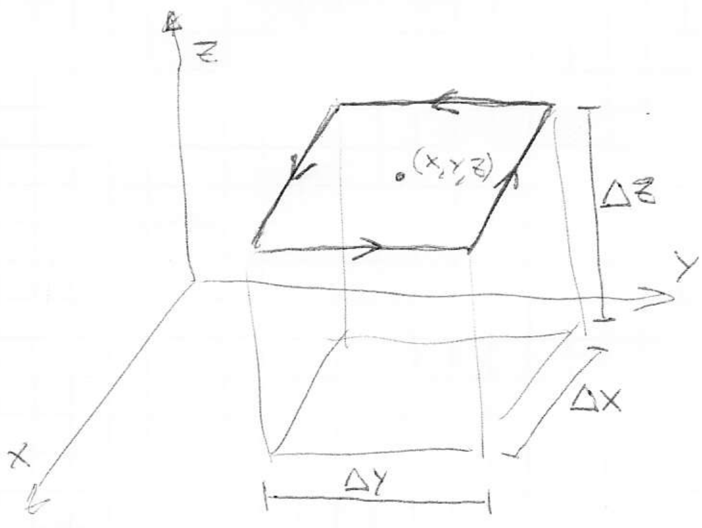
PATH INDEPENDENCE MEANS THAT THE RESULT OF THE INTEGRAL DEPENDS ONLY ON THE INITIAL AND FINAL POINTS, INDEPENDENTLY OF THE PATH OF INTEGRATION.

CHECKING THIS CONDITION OF PATH INDEPENDENCE IS OF INTEREST, BUT NOT PRACTICALLY POSSIBLE USING

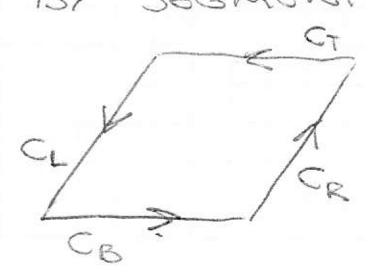
$$\oint_C \bar{F} \cdot \hat{t}\, dc = 0$$

SINCE WE WOULD HAVE TO INTEGRATE OVER AN INFINITE NUMBER OF PATHS TO PROVE THAT EXPRESSION TRUE ... BUT WHAT IF WE CONSIDER AN EVER-DECREASING VOLUME?

LET'S CONSIDER A SMALL CUBE, IN CARTESIAN COORDINATES, AND THE CIRCULATION OVER THE TOP FACE OF THAT CUBE (PERIMETER). WE WILL USE THE RIGHT HAND RULE FOR DIRECTION OF CIRCULATION, TAKING INTO ACCOUNT THAT THE NORMAL VECTOR $\hat{n}$ TO THE SURFACE POINTS OUTWARDLY OF THE VOLUME.

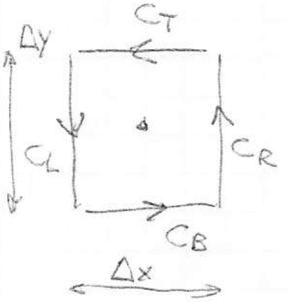


WE CAN NOW INTEGRATE OVER THIS PATH BY SEGMENTS,



AND SINCE THE RECTANGLE IS VERY SMALL, WE CAN APPROXIMATE THE INTEGRAL OVER EACH SEGMENT BY EVALUATING $\bar{F}\cdot\hat{t}$ AT THE SEGMENT'S CENTER POINT AND MULTIPLYING BY THE SEGMENT LENGHT.

SO AT C_B,



$$\int_{C_B} \bar{F}\cdot\hat{t}\, ds = \int_{C_B} F_x dx \simeq F_x\left(x, y-\frac{\Delta y}{2}, z\right)\Delta x$$

AND AT C_T,

$$\int_{C_T} F_x dx \simeq -F_x\left(x, y+\frac{\Delta y}{2}, z\right)\Delta x$$

ADDING THE COMPONENTS ON THE X-DIRECTION (C_B+C_T),

$$\int_{C_B+C_T} F_x dx = F_x\left(x, y-\frac{\Delta y}{2}, z\right)\Delta x - F_x\left(x, y+\frac{\Delta y}{2}, z\right)\Delta x$$

AND DIVIDING NOW BY THE AREA OF THE RECTANGLE,

$$\Delta S = \Delta x\, \Delta y$$

$$\frac{1}{\Delta S}\int_{C_B+C_T} F_x dx = \frac{1}{\Delta S}\int_{C_B+C_T} \bar{F}\cdot\hat{t}\, d_c \simeq \underbrace{-\left(\frac{F_x\left(x, y+\frac{\Delta y}{2}, z\right) - F_x\left(x, y-\frac{\Delta y}{2}, z\right)}{\Delta y}\right)}_{}$$

WHEN TAKING LIMIT FOR $\Delta S \to 0 \Rightarrow -\dfrac{\partial F_x}{\partial y}$

DOING THE SAME PROCESS WITH C_R AND C_L,

$$\frac{1}{\Delta S}\int_{C_R+C_L} \bar{F}\cdot\hat{t}\,ds \simeq \frac{F_Y\left(x+\frac{\Delta x}{2},y,z\right)-F_Y\left(x-\frac{\Delta x}{2},y,z\right)}{\Delta x} \Rightarrow \frac{\partial F_Y}{\partial x}$$

SO CONSIDERING THE FOUR SIDES OF THE CURVE,

$$\lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_{C_T+C_B+C_L+C_R} \bar{F}\cdot\hat{t}\,dc = \frac{\partial F_Y}{\partial x}-\frac{\partial F_X}{\partial Y} \quad (\hat{k} \text{ NORMAL CASE})$$

IN CARTESIAN COORDINATES THERE ARE TWO OTHER ORTHOGONAL DIRECTIONS (FOR 3D), WHICH ALL CONSIDERED RESULTS IN:

XY PLANE ($\hat{k}$ NORMAL) $\quad \dfrac{\partial F_Y}{\partial x}-\dfrac{\partial F_X}{\partial Y}$

XZ PLANE ($\hat{J}$ NORMAL) $\quad \dfrac{\partial F_X}{\partial z}-\dfrac{\partial F_Z}{\partial x}$

YZ PLANE ($\hat{\imath}$ NORMAL) $\quad \dfrac{\partial F_Z}{\partial Y}-\dfrac{\partial F_Y}{\partial z}$

WE DEFINE A NOTATION

$$\boxed{\lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint \bar{F}\cdot\hat{t}\,dc = \hat{n}\cdot \text{CURL}\,\bar{F} = \hat{n}\cdot\nabla\times\bar{F}}$$

WHICH IS THE DEFINITION OF CURL. AND SINCE $\bar{F} = F_X\hat{\imath} + F_Y\hat{J} + F_Z\hat{k}$

$$\boxed{\nabla\times\bar{F} = \hat{\imath}\left(\frac{\partial F_Z}{\partial Y}-\frac{\partial F_Y}{\partial z}\right)+\hat{J}\left(\frac{\partial F_X}{\partial z}-\frac{\partial F_Z}{\partial x}\right)+\hat{k}\left(\frac{\partial F_Y}{\partial x}-\frac{\partial F_X}{\partial Y}\right)}$$

IN CARTESIAN COORDINATES.

WE CAN USE THE MNEMONIC

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

CURL IS THE LIMIT OF CIRCULATION TO AN AREA

IF WE CONSIDER AN ELECTROSTATIC FIELD $\bar{E}$, WE KNOW THAT SINCE THE FIELD IS CONSERVATIVE,

$$\oint_c \bar{E} \cdot \hat{t}\, dc = 0$$ FOR ANY PATH.

IN CARTESIAN COORDINATES, TAKING EACH COMPONENT,

$$\left.\begin{aligned} \hat{n} = \hat{i} \quad &\Rightarrow \quad \nabla \times \bar{E}\Big|_x = 0 \\ \hat{n} = \hat{j} \quad &\Rightarrow \quad \nabla \times \bar{E}\Big|_y = 0 \\ \hat{n} = \hat{k} \quad &\Rightarrow \quad \nabla \times \bar{E}\Big|_z = 0 \end{aligned}\right\}$$

SINCE ALL COMPONENTS ARE ZERO, WE CONCLUDE THAT

$$\boxed{\nabla \times \bar{E} = 0}$$

FOR AN ELECTROSTATIC FIELD.