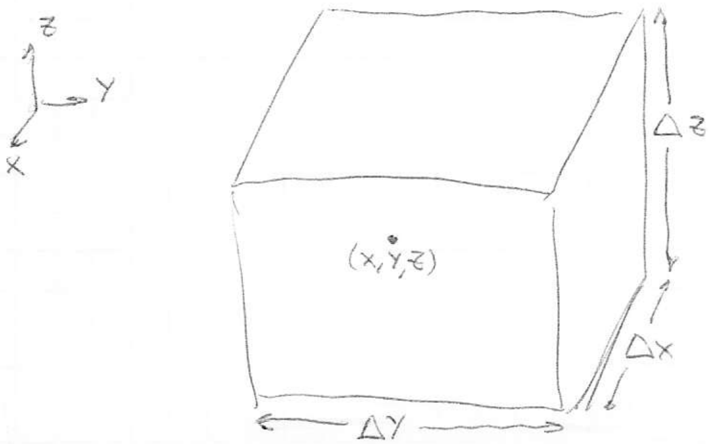
DIVERGENCE

CONSIDER A SMALL CUBOID VOLUME ALIGNED TO THE CARTESIAN COORDINATE AXES, CENTERED ON THE POINT (X, Y, Z)



THE ENCLOSED VOLUME IS

$$\Delta V = \Delta X \, \Delta Y \, \Delta Z$$

WE WANT TO CALCULATE THE FLUX OF A VECTOR FUNCTION OVER THE SURFACE OF OUR VOLUME, BY ADDING THE CONTRIBUTION OF EACH SIDE.

FROM OUR DEFINITION OF FLUX,

$$\Phi_S = \int_S \bar{F} \cdot \hat{n} \, ds$$

FURTHER,

$$\bar{F} \cdot \hat{i} = F_x \qquad \text{OR} \qquad \bar{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$
$$\bar{F} \cdot \hat{j} = F_y$$
$$\bar{F} \cdot \hat{k} = F_z$$

CONSIDERING THE FLUX OF $\bar{F}$ ON THE <u>FRONT</u> FACE,

$$\Phi_{S_1} = \int_{S_1} \bar{F} \cdot \hat{n} \, ds \equiv \int_{S_1} \bar{F} \cdot \hat{i} \, ds \equiv \int_{S_1} F_x \, ds = F_x\left(X + \frac{\Delta X}{2}, Y, Z\right) \Delta Y \, \Delta Z$$

IF THE SURFACE S_1 IS SMALL ENOUGH (I.E. F_x APPROXIMATED AS CONSTANT OVER S_1).

CONSIDERING NOW THE <u>BACK</u> FACE,

$$\Phi_{S_2} = \int_{S_2} \bar{F}\cdot(-\hat{\imath})\,ds \equiv -\int_{S_2} f_x\,ds = -F_x\left(x-\frac{\Delta x}{2}, y, z\right)\Delta y\,\Delta z$$

THE TOTAL CONTRIBUTION ALONG THE X-AXIS WILL THEN BE

$$\Phi_{S_1+S_2} = \left[F_x\left(x+\frac{\Delta x}{2}, y, z\right) - F_x\left(x+\frac{\Delta x}{2}, y, z\right)\right]\Delta y\,\Delta z = \int_{S_1 S_2} \bar{F}\cdot\hat{n}\,ds$$

DIVIDING BY $\Delta V = \Delta x\ \Delta y\ \Delta z$

$$\frac{1}{\Delta V}\int_{S_1+S_2} \bar{F}\cdot\hat{n}\,ds = \frac{F_x\left(x+\frac{\Delta x}{2}, y, z\right) - F_x\left(x-\frac{\Delta x}{2}, y, z\right)}{\Delta x}$$

'SHRINKING' THE VOLUME NOW, WE APPLY THE LIMIT WHEN $\Delta V \to 0$, WHICH IMPLIES $\Delta x \to 0$,

$$\lim_{\Delta V\to 0} \frac{1}{\Delta V}\int_{S_1+S_2} \bar{F}\cdot\hat{n}\,ds = \frac{\partial F_x}{\partial x}$$

IF WE PROCEED SIMILARLY WITH THE OTHER SURFACES, WE <u>ARRIVE</u> AT:

$$\boxed{\lim_{\Delta V\to 0} \frac{1}{\Delta V}\oint_S \bar{F}\cdot\hat{n}\,ds} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

WHICH WE CALL 'DIVERGENCE OF $\bar{F}$', AND DENOTE AS

$$\nabla\cdot F \qquad \text{OR} \qquad \text{DIV } \bar{F}$$

$$\boxed{\nabla\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}}$$

EXPRESSION OF <u>D</u>IVERGENCE OF F FOR <u>CARTESIAN</u> <u>COORDINATES</u>.