ECE5340/6340: Homework 4
ANSWER KEY

Department of Electrical and Computer Engineering
University of Utah, Salt Lake City, Utah
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1. Re-derive the error bound for the midpoint rule and show that it satisfies

$$\epsilon \approx -\frac{1}{24} f^{(2)}(c)\Delta x^3 \ .$$

This is in the notes.

2. For a single trapezoid of width Δx and heights $f(a)$ and $f(b)$, the area is given as simply $A = \Delta x(f(a)+f(b))/2$. From this knowledge, show that if we subdivide the interval $[a, b]$ into n subintervals, then the area under the function $f(x)$ may be approximated as

$$A' = \Delta x \left[\frac{f(a)}{2} + \frac{f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right] \ ,$$

where $x_i = a + i\Delta x$.

<u>SOLUTION</u>: For a series of trapezoids defined along the samples placed at x_i, the area of the ith trapezoid is simply

$$A_i = \frac{\Delta x}{2} \Big[f(x_i) + f(x_{i+1}) \Big] \ .$$

Now let us add up the area of n trapezoids:

$$\sum_{i=1}^{n} A_i = \frac{\Delta x}{2} \Big[f(x_0) + f(x_1) \Big] + \frac{\Delta x}{2} \Big[f(x_1) + f(x_2) \Big] + \frac{\Delta x}{2} \Big[f(x_2) + f(x_3) \Big] + \cdots + \frac{\Delta x}{2} \Big[f(x_{n-2}) + f(x_{n-1}) \Big] + \frac{\Delta x}{2} \Big[f(x_{n-1}) + f(x_n) \Big] \ .$$

Next, we note that $x_0 = a$ and $x_n = b$ to find

$$\sum_{i=1}^{n} A_i = \frac{\Delta x}{2} \Big[f(a) + f(b) \Big] + \frac{\Delta x}{2} \Big[2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-2}) + f(x_{n-1}) \Big] \ .$$

Simplify this expression to arrive at

$$\sum_{i=1}^{n} A_i = A' = \boxed{\Delta x \left[\frac{f(a)}{2} + \frac{f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right]} \ .$$

3. Derive the error bound for the trapezoidal rule and show that it satisfies

$$\epsilon \approx \frac{1}{12} f^{(2)}(c) \Delta x^3 ,$$

where $c = (a+b)/2$ is the midpoint between the interval $[a, b]$.

SOLUTION: Let us begin by defining two Taylor series expansions for area under a curve. The first series is defined with respect to the point $x = a$, so let's call this the "left" area:

$$A_L = f(a)\Delta x + \frac{1}{2!} f'(a)\Delta x^2 + \frac{1}{3!} f^{(2)}(a)\Delta x^3 + \cdots$$

Now let us also define the "right" area in the same fashion, but around the point $x = b$.

$$A_R = f(b)\Delta x - \frac{1}{2!} f'(b)\Delta x^2 + \frac{1}{3!} f^{(2)}(b)\Delta x^3 - \cdots$$

Remember that both of these areas are just fancy ways of calculating the EXACT area under the curve, which means that $A_L = A_R = A$. This means if we add them together and divide by 2, we still have the true area:

$$\begin{aligned} A &= \frac{A_L + A_R}{2} \\ &= \frac{1}{2}\Big[f(a)\Delta x + f(b)\Delta x + \frac{1}{2} f'(a)\Delta x^2 - \frac{1}{2} f'(b)\Delta x^2 + \frac{1}{6} f^{(2)}(a)\Delta x^3 + \frac{1}{6} f^{(2)}(b)\Delta x^3 + \cdots \Big] \\ &= \frac{1}{2} f(a)\Delta x + \frac{1}{2} f(b)\Delta x + \frac{1}{4} f'(a)\Delta x^2 - \frac{1}{4} f'(b)\Delta x^2 + \frac{1}{12} f^{(2)}(a)\Delta x^3 + \frac{1}{12} f^{(2)}(b)\Delta x^3 + \cdots . \end{aligned}$$

Next, we note that the first two terms in this series represent an approximate area due to a trapezoid.

$$A' = \frac{1}{2} f(a)\Delta x + \frac{1}{2} f(b)\Delta x .$$

The total error in the approximate area is therefore

$$\begin{aligned} \epsilon &= A' - A \\ &= -\frac{1}{4} f'(a)\Delta x^2 + \frac{1}{4} f'(b)\Delta x^2 - \frac{1}{12} f^{(2)}(a)\Delta x^3 - \frac{1}{12} f^{(2)}(b)\Delta x^3 + \cdots . \end{aligned}$$

The tricky part is to recognize an approximate derivative buried in this expression. Begin by noting that $b = a + \Delta x$ and defining the midpoint $c = (a+b)/2$. We then note that for small values of Δx, the central difference formula allows us to write

$$\begin{aligned} f^{(2)}(c) &\approx \frac{f'(c + \Delta x/2) - f'(c - \Delta x/2)}{\Delta x} \\ &= \frac{f'(b) - f'(a)}{\Delta x} \end{aligned}$$

This allows us to make the following replacement:

$$-\frac{1}{4} f'(a)\Delta x^2 + \frac{1}{4} f'(b)\Delta x^2 \approx \frac{1}{4}\Delta x^3 f^{(2)}(c)$$

The next trick is to perform another approximation under the assumption of very small values for Δx:

$$f^{(2)}(c) \approx \frac{1}{2} f^{(2)}(a) + \frac{1}{2} f^{(2)}(b) .$$

Note that this is simply implying that the value of a function f is approximately the average of the values nearby. This allows us to insert another replacement:

$$-\frac{1}{12}f^{(2)}(a)\Delta x^3 - \frac{1}{12}f^{(2)}(b)\Delta x^3 \approx -\frac{1}{6}f^{(2)}(c)\Delta x^3$$

Putting it all together, we finally have

$$\begin{aligned}\epsilon &\approx \frac{1}{4}\Delta x^3 f^{(2)}(c) - \frac{1}{6}f^{(2)}(c)\Delta x^3\\ &= \boxed{\frac{1}{12}\Delta x^3 f^{(2)}(c)}\ .\end{aligned}$$

4. Write a Matlab code that calculates the numerical approximation to a definite integral using the left-point rule. Repeat for the midpoint rule, trapezoidal rule, and Simpson's rule. You will use these in the following sections (Note: Matlab has a built-in functions like "trapz" that implement numerical integration algorithms. You can compare against these functions for debugging purposes, but be sure to write your own code).

   SOLUTION: See m-files.

5. Consider the definite integral

$$A_k = \int_1^2 f_k(x)\,dx\ .$$

   Solve for A_k using the following set of functions:

$$\begin{aligned}f_1(x) &= x\\ f_2(x) &= x^2\\ f_3(x) &= x^3\\ f_4(x) &= x^4\\ f_5(x) &= \sin x\end{aligned}$$

   These functions will serve as our test functions for computing numerical integrals. Fill in your analytical solutions using Table 1 at the end of this document.

   SOLUTION: See tables.

6. Use the error bound formulas to calculate the expected error for both the Trapezoidal rule and Simpson's rule that should arise from using $n = 10$ subintervals. Compare these errors against the true errors by using your Matlab programs. Fill in your solutions using Table 2.

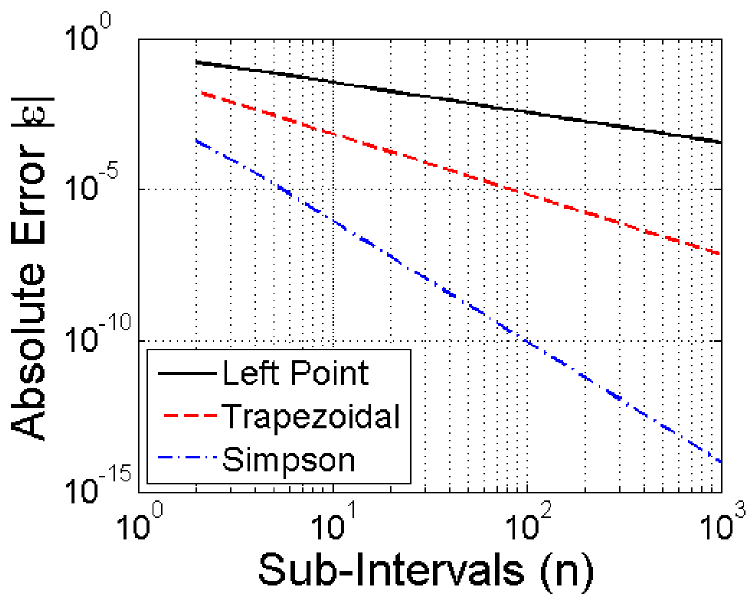
   SOLUTION: See tables. Note that there is some ambiguity with Simpson's rule. References from other sources will tend to express the error bound with a factor of 1/90 rather than 1/180. This is because they compute the total error for a single parabolic subinterval. In practice, it is generally more intuitive to think of each little Δx region as a subinterval, which is actually one-half of the total parabolic section. This obviously changes things by a factor of 2. However, the exact error is not terribly important in the real world. Rather, the thing that really matters is the change in error with respect to the number of fevals. So any answer within the general ballpark is good enough for this problem.

7. Some important mathematical functions are very difficult to calculate analytically. For example, consider the *error function* defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \ .$$

Calculate the absolute error $|\epsilon|$ of a numerical approximation to erf(x) as a function of increasing subintervals using $[a, b] = [0, 1]$. Compare the left-point rule, the trapezoid rule, and Simpson's rule against the exact value (hint: use Matlab's built-in function "erf" to find the true area). Plot your results on a log-log scale from $n = 2$ subintervals up to $n = 1000$.

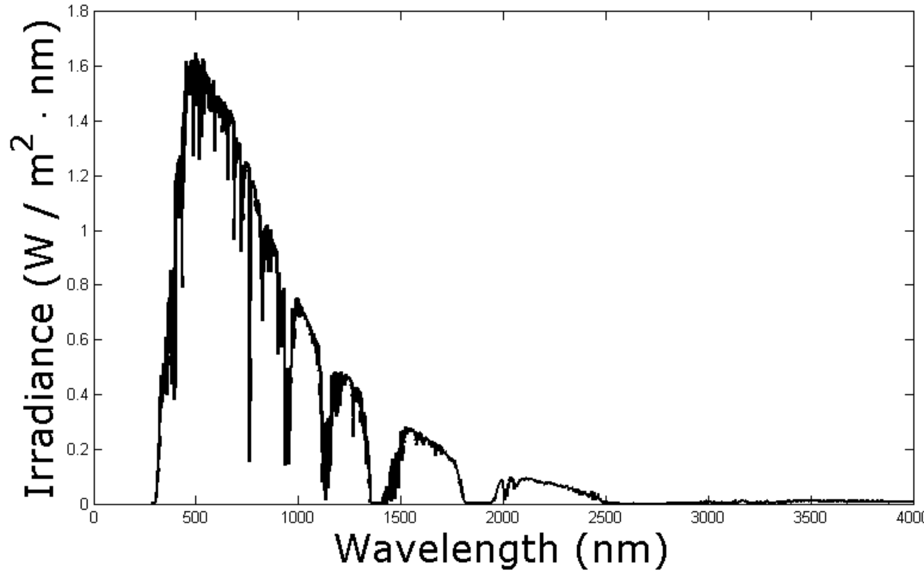
SOLUTION: See figure below:



Note that a plot like this would not make any sense if we tried to use total parabolic sections as the subintervals for Simpson's rule. The reason is because the number of subintervals n would not represent the same thing between integration methods. For this reason, I tend to dislike the alternative conventions that people try to push when introducing Simpson's rule.

8. Download the AM 1.0 spectral irradiance data contained in the Matlab file *solarData.mat* from the website. This data represents the instantaneous influx of solar radiation to an observer on Earth with the sun directly overhead. Use numerical integration on the data set to calculate how much energy the sun is delivering to us. Note that the wavelength units are given in nanometers and the power flux is given in units of W/m^2·nm (Be careful! The wavelength data is not uniformly spaced).

SOLUTION: The AM 1.0 data is shown below:



Using trapezoidal integration, the total solar influx is approximately 1.0 kW/m^2.

9. ECE 6340 Only: Analytically solve the integral

$$A_k = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_k(x, y)\, dy dx \ ,$$

over the domain $[x_1, x_2] = [0, 1]$ and $[y_1, y_2] = [0, 3]$ for the following set of functions:

$$f_1(x, y) = xy$$
$$f_2(x, y) = x^4 y^4$$
$$f_3(x, y) = e^{xy}$$

NOTE: f_3 is not a separable function, and therefore cannot be solved by hand. So just use a value of $A_3 \approx 8.258$ to verify your code.

10. ECE 6340 Only: Write a Matlab program that calculates the numerical integral of a 2D function by applying the trapezoidal rule. Compare your numerical results against the exact values by filling in Table 3. Use $\Delta x = \Delta y = 0.01$ for setting up your integration.

$f(x)$	Analytical Solution	Trapezoidal Rule	Simpson's Rule
x	1.5	1.5	1.5
x^2	2.3333	2.335	2.3333
x^3	3.75	3.7575	3.75
x^4	6.2	6.2233	6.2
$\cos x$	0.9564	0.9557	0.9564

Table 1: 1-D numerical integration summary using $n = 10$ on the interval $[a, b] = [1, 2]$.

$f(x)$	Trapezoidal Error (Expected)	Trapezoidal Error (Observed)	Simpson's Error (Expected)	Simpson's Error (Observed)
x	0	0	0	0
x^2	0.0017	0.0017	0	0
x^3	0.0075	0.0075	0	0
x^4	0.0235	0.0233	0.1333×10^{-4}	0.1333×10^{-4}
$\cos x$	-0.0008	-0.0008	0.0055×10^{-4}	0.0053×10^{-4}

Table 2: Error summary.

$f(x, y)$	Analytical Solution	Trapezoidal Rule
xy	2.25	2.25
x^4y^4	9.72	9.7218
e^{xy}	≈ 8.258	8.2583

Table 3: 2-D integration summary.