ECE5340/6340: Homework 5
Finite Difference Method

Department of Electrical and Computer Engineering
University of Utah, Salt Lake City, Utah
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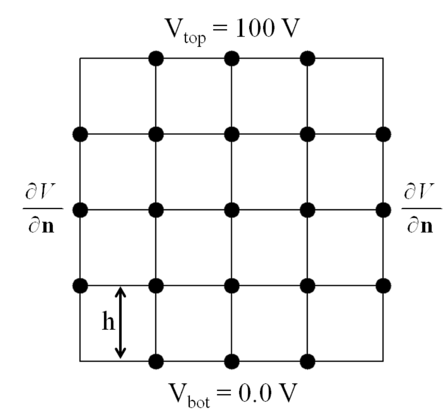
Write your section (ECE5340 or ECE6340) by your name. Turn in a printed copy containing the problem solutions, plots, and the code used to generate them. Remember to comment and format the code so it is legible to the graders. Label the plots appropriately, including units for each axis and for the values plotted. Assume all units to be SI units unless stated differently. Due Wednesday 2/15 BEFORE class begins.

ASSIGNMENT

1. Starting from Gauss's law in point form, derive the Poisson equation for voltage potential. Assume that the dielectric function ϵ is a constant throughout all space.
2. Using the five-point star, discretize the Poisson equation along a rectangular grid. Solve for $V(i,j)$ in terms of the neighboring points.
3. Consider the discretized region of voltage samples shown below:



   Assume that the top voltage samples are given by $V_{top} = 100$ V and that the bottom samples are given by $V_{bot} = 0$ V. The left and right boundaries are Neumann boundaries with a derivative of 25 V/m with respect to the *outward unit normal.*

   Using a grid spacing of $h = 0.25$ m, derive the matrix-vector equation $\mathbf{Ax} = \mathbf{b}$, where the vector $\mathbf{x}$ contains the samples of voltage potential along the grid. Write it out explicitly, showing all elements in the system, and solve for the voltage samples in $\mathbf{x}$.

- Comment on the size of $\mathbf{A}$. If we were to double the grid resolution (ie, 10 $\times$ 10 grid samples instead of 5 $\times$ 5), what would the size of $\mathbf{A}$ be?
- Approximately how many voltage samples would it take ($N \times N$) before your computer would melt down trying to solve the system directly?
- How many elements in $\mathbf{A}$ are zero? How many are non-zero? If we doubled the grid resolution, how many elements would be zero and how many would be non-zero? What does this tell you about the *sparse* nature of $\mathbf{A}$?

4. Simulate the voltage potential due to a parallel-plate capacitor by applying FDM with SOR in two dimensions. Use a plate width of $W = 1.0$ meters and a plate separation of $d = 0.2$ meters. Fix the top plate to a voltage of $+1.0$ V and the bottom plate to -1.0 V. Set the exterior boundaries of the simulation to a size of 2.0 meters (width) $\times$ 1.0 meters (height) using a grid spacing of $h = 0.01$ m.

   - Plot the voltage potential on a 2D image plot to show that your simulation worked properly.
   - Calculate the electric field components using the numerical approximation to the gradient of V. Find $|\mathbf{E}| = \sqrt{E_x^2 + E_y^2}$ and plot this on a 2D image. Where are the electric fields mostly concentrated?
   - How much memory would be required if we tried to directly fill the system matrix for this simulation? How did SOR allow us to solve this problem much more efficiently?
   - How many iterations of SOR were required to complete the simulation and arrive at a solution? What is the ideal value of ω that you used?
   - <u>ECE 6340 ONLY</u>: Try re-running the simulation at several different values of ω and comment on differences you observe. Plot the number of iterations required to reach convergence as a function of ω. Use $\omega = [1.5 : 0.05 : 1.95]$. Be sure to note your cutoff point with the residual as your convergence criterion.

5. Use Gauss's law to calculate the capacitance per unit length of your simulated capacitor. Compare against the common analytical solution using

$$C' = \frac{\epsilon_0 W}{d} \quad \text{(F/m)} ,$$

   where W is the width of the plates, d is the separation distance, and ϵ_0 is the permittivity of free space.

   - How well does your simulated value agree with the analytical computation?
   - Which value do you think is the more accurate solution? The analytical or the simulated? Why?
   - What assumptions does the analytical expression make (hint: think back to your previous courses in electromagnetics when you first derived this equation)? What errors exist in the simulation model?