How to Compute Characteristic Impedance from FDM Simulation

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First, we note the definition of characteristic impedance for a transmission line:

$$Z = \sqrt{\frac{L}{C}} \ , \tag{1}$$

where L is the inductance per unit length and C is the capacitance per unit length. Next, we recall that the phase velocity satisfies

$$v_p = \frac{1}{\sqrt{LC}} \ . \tag{2}$$

We now define the free-space equivalent of these parameters using

$$Z_0 = \sqrt{\frac{L_0}{C_0}} \ , \tag{3}$$

$$v_{p0} = \frac{1}{\sqrt{L_0 C_0}} \ . \tag{4}$$

In this context, the subscript zero simply denotes the identical system without any embedded dielectric materials. Because neither system has any embedded magnetic materials, we can also assume that $L_0 = L$. Also, it is known that for a system with no insulation between the conductors, the phase velocity satisfies $v_{p0} = c_0 = 2.996 \times 10^8$ m/s.

We now rewrite the original characteristic impedance into a special form:

$$\begin{aligned} Z &= \sqrt{\frac{L}{C}} \\ &= \sqrt{\frac{LC_0}{CC_0}} \\ &= \sqrt{LC_0}\sqrt{\frac{1}{CC_0}} \\ &= \frac{1}{v_{p0}\sqrt{CC_0}} \end{aligned} \tag{5}$$

The only unknowns in this expression are the capacitances per unit length. However, from the definition of capacitance, we know that

$$C = \frac{q}{V_0} \ , \tag{6}$$

where q is the charge stored on the capacitor and V_0 is the static voltage potential of the system. In an FDM simulation, V is a known function that can be imposed arbitrarily. The stored charge can also be found by applying Gauss's law. Writing it out in integral form, this is

$$\oiint_S \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} = q \ , \tag{7}$$

where the surface S encloses the charge and $d\mathbf{S}$ is the differential unit normal to the surface. Since we're dealing strictly with 2D simulations, we can rewrite this explicitly as

$$\oint_C \epsilon(x, y)\mathbf{E}(x, y) \cdot d\boldsymbol{\ell} = q \ , \tag{8}$$

where $d\boldsymbol{\ell}$ is a differential piece of the contour C with respect to the outward unit normal vector. Since both of the quantities within the integral are known outputs from our FDM simulation, it is just a matter of numerical integration to produce a calculation for q.

Note that for our purposes, the simplest choice for the contour C will just be a square that encloses one of the capacitor plates. The entire integral can then be subdivided into four sections, each representing one side of the square. The only really tricky part is properly defining $d\boldsymbol{\ell}$, which is summarized here:

$$\begin{aligned} d\boldsymbol{\ell} &= +h\hat{\mathbf{y}} \qquad &&\text{(top)} \\ &= -h\hat{\mathbf{y}} \qquad &&\text{(bottom)} \\ &= +h\hat{\mathbf{x}} \qquad &&\text{(right)} \\ &= -h\hat{\mathbf{x}} \qquad &&\text{(left)} \end{aligned}$$

In this form, the dot product essentially singles out one of the field components for $\mathbf{E}$, thus making integration much simpler.