ECE5340/6340: Homework 5
ANSWER KEY

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Write your section (ECE5340 or ECE6340) by your name. Turn in a printed copy containing the problem solutions, plots, and the code used to generate them. Remember to comment and format the code so it is legible to the graders. Label the plots appropriately, including units for each axis and for the values plotted. Assume all units to be SI units unless stated differently. Due Wednesday 2/15 BEFORE class begins.

ASSIGNMENT

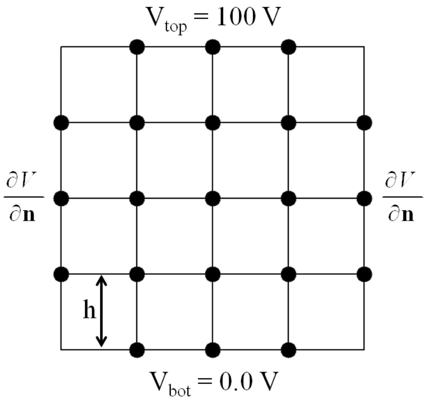
1. Starting from Gauss's law in point form, derive the Poisson equation for voltage potential. Assume that the dielectric function ϵ is a constant throughout all space.

   This is in the introduction to the notes. Simply copy and paste and show that you understand what is going on.

2. Using the five-point star, discretize the Poisson equation along a rectangular grid. Solve for $V(i, j)$ in terms of the neighboring points.

   Again, this is likewise in the lecture notes. Simply copy and paste and show that you understand what is going on.

3. Consider the discretized region of voltage samples shown below:


$V_{top} = 100$ V
$\frac{\partial V}{\partial \mathbf{n}}$
$\frac{\partial V}{\partial \mathbf{n}}$
h
$V_{bot} = 0.0$ V

Assume that the top voltage samples are given by $V_{top} = 100$ V and that the bottom samples are given by $V_{bot} = 0$ V. The left and right boundaries are Neumann boundaries with a derivative of 25 V/m with respect to the *outward unit normal*.

Using a grid spacing of $h = 0.25$ m, derive the matrix-vector equation $\mathbf{Ax} = \mathbf{b}$, where the vector $\mathbf{x}$ contains the samples of voltage potential along the grid. Write it out explicitly, showing all elements in the system, and solve for the voltage samples in $\mathbf{x}$.

This is where things begin to get tedious, but it is a good exercise for understanding the essence of FDM. We begin by enumerating all of the voltage samples in some convention that makes sense. I will stick to the numbering used in the lecture notes by starting from the bottom left and scanning upward. The result is a massive series of linear equations that relate all of the voltages together.

Starting with the Dirichlet boundary conditions at the bottom and top, we have six equations that are essentially "given" to us. There are

$$\begin{aligned} V_1 &= 0 \text{ V} \\ V_2 &= 0 \text{ V} \\ V_3 &= 0 \text{ V} \\ V_{19} &= 100 \text{ V} \\ V_{20} &= 100 \text{ V} \\ V_{21} &= 100 \text{ V} \end{aligned}$$

The Neumann boundaries are a little trickier this time because the derivatives are nonzero. Using a forward finite-difference along the left boundary, we have

$$\begin{aligned} V_4 - V_5 &= 25h \\ V_9 - V_{10} &= 25h \\ V_{14} - V_{15} &= 25h \end{aligned}$$

Remember that the convention is "boundary minus inner" in order to get the proper derivatives. So repeating along the right boundary gives us

$$\begin{aligned} V_8 - V_7 &= 25h \\ V_{13} - V_{12} &= 25h \\ V_{18} - V_{18} &= 25h \end{aligned}$$

Finally, we apply the five-point star at all of the inner points. This gives us nine more equations.

$$\begin{aligned}
-4V_5 + V_1 + V_6 + V_4 + V_{10} &= 0\\
-4V_6 + V_2 + V_7 + V_5 + V_{11} &= 0\\
-4V_7 + V_3 + V_8 + V_6 + V_{12} &= 0\\
-4V_{10} + V_5 + V_{11} + V_{15} + V_9 &= 0\\
-4V_{11} + V_6 + V_{12} + V_{16} + V_{10} &= 0\\
-4V_{12} + V_7 + V_{13} + V_{17} + V_{11} &= 0\\
-4V_{15} + V_{10} + V_{16} + V_{19} + V_{14} &= 0\\
-4V_{16} + V_{11} + V_{17} + V_{20} + V_{15} &= 0\\
-4V_{17} + V_{12} + V_{18} + V_{21} + V_{16} &= 0
\end{aligned}$$

Now we are ready to fill the system matrix (copy/paste really comes in handy here).

$$\left[\begin{array}{ccccccccccccccccccccc}
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & \mathbf{1} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & 0 & \mathbf{1}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}
\end{array}\right]$$

The **b**-vector is then given as

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6.25 \\ 0 \\ 0 \\ 0 \\ 6.25 \\ 6.25 \\ 0 \\ 0 \\ 0 \\ 6.25 \\ 6.25 \\ 0 \\ 0 \\ 0 \\ 6.25 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

Finally, solve for **x** to find

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 38.0435 \\ 31.7935 \\ 30.1630 \\ 31.7935 \\ 38.0435 \\ 65.2174 \\ 58.9674 \\ 57.0652 \\ 58.9674 \\ 65.2174 \\ 88.0435 \\ 81.7935 \\ 80.1630 \\ 81.7935 \\ 88.0435 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

- Comment on the size of **A**. If we were to double the grid resolution (ie, 10 × 10 grid samples instead of 5 × 5), what would the size of **A** be?

A is currently 21×21 elements. If we included the corners of the box, then it would have been

25×25 elements. If the simulation domain were 10×10 voltage samples, then there would be 100 total unknowns. This would force **A** to be 100 ×100 elements, or 10,000 total.

- How many elements in **A** are zero? How many are non-zero? If we doubled the grid resolution, how many elements would be zero and how many would be non-zero? What does this tell you about the *sparse* nature of **A**?

Total number of elements in the system matrix is 441. Total nonzero elements is only 63, or 14% of the total in **A**. If we double the grid density to 10×10, then the system will have (including the corners) 100 unknowns and 10,000 elements in **A**. At most, each row in **A** can only have 5 unknowns. Thus, the total nonzero elements is 500 at most, or only 5%. Thus, as **A** grows larger, the fraction of nonzero elements approaches zero.

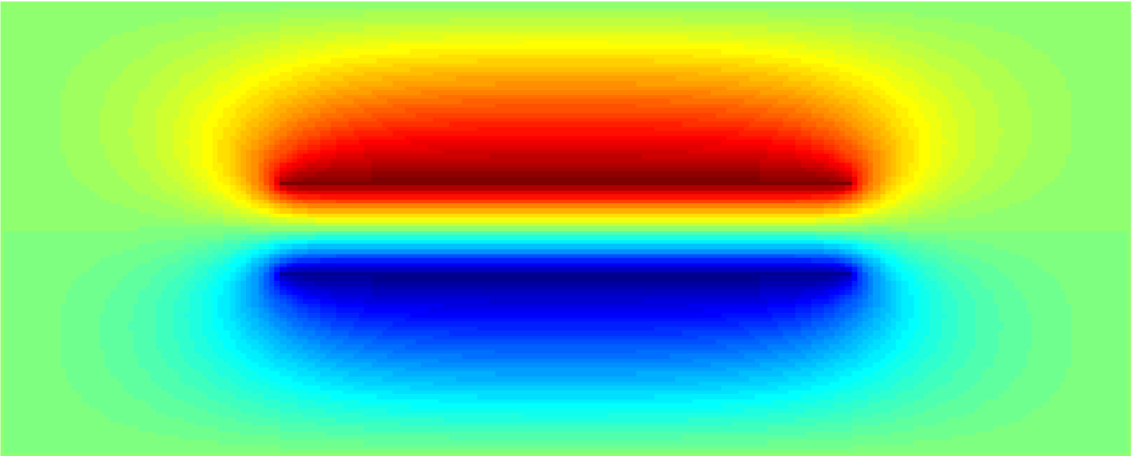
- Approximately how many voltage samples would it take ($N \times N$) before you think your computer would melt down trying to solve the system directly?

Individual answers may vary, but the thought process should look like this:

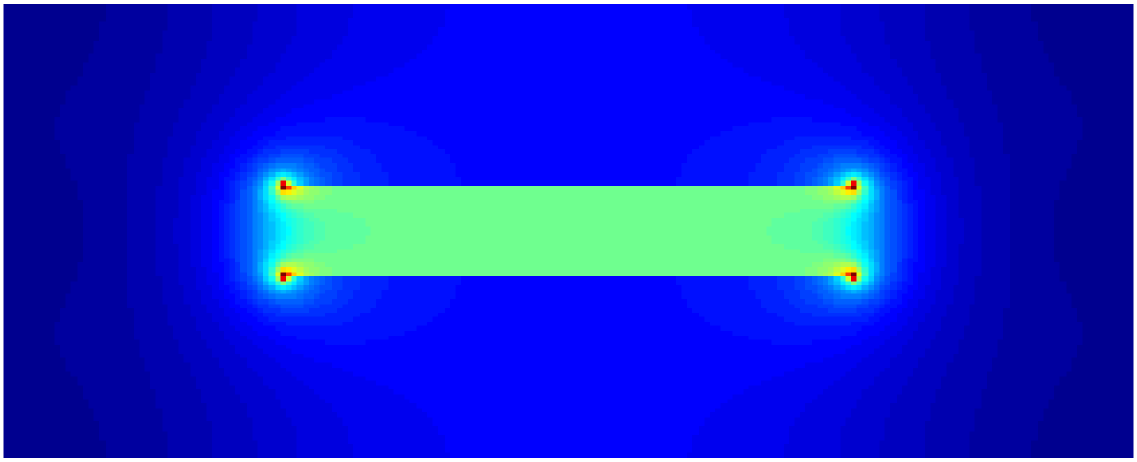
Let us assume 4GB of RAM for the average desktop. Let us also assume it takes one byte of memory for each elements in **A**. That means **A** can be roughly 63,000×63,000 elements, which corresponds to a voltage grid of only 251×251 samples. Note that this does NOT include the extra cost of storing double-precision floating point variables or the memory cost of inverting a matrix.

4. Simulate the voltage potential due to a parallel-plate capacitor by applying FDM with SOR in two dimensions. Use a plate width of $W = 1.0$ meters and a plate separation of $d = 0.2$ meters. Fix the top plate to a voltage of $+1.0$ V and the bottom plate to -1.0 V. Set the exterior boundaries of the simulation to a size of 2.0 meters (width) × 1.0 meters (height) using a grid spacing of $h = 0.01$ m.

   - Plot the voltage potential on a 2D image plot to show that your simulation worked properly.



- Calculate the electric field components using the numerical approximation to the gradient of V. Find $|\mathbf{E}| = \sqrt{E_x^2 + E_y^2}$ and plot this on a 2D image. Where are the electric fields mostly concentrated?

  Notice how most of the fields are packed between the capacitor plates, as we should expect.

- How much memory would be required if we tried to directly fill the system matrix for this simulation? How did SOR allow us to solve this problem much more efficiently?

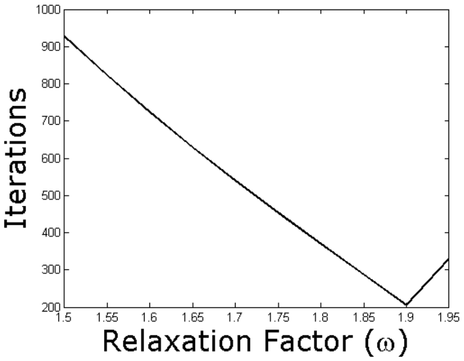
  The simulation domain is 201×101 samples, giving a total number of 20,301 unknowns. If we tried to directly fill the system matrix, this would require over 400 million elements to get stored in memory. By applying SOR, we only needed to store the voltage samples plus some overhead. This makes the difference between only tens of thousands of values being stored rather than hundreds of millions.

- How many iterations of SOR were required to complete the simulation and arrive at a solution? What is the ideal value of ω that you used?

  Individual answers can vary. Using $\omega = 1.952$, the total number of iterations I needed was 344.

- <u>ECE 6340 ONLY</u>: Try re-running the simulation at several different values of ω and comment on differences you observe. Plot the number of iterations required to reach convergence as a function of ω. Use $\omega = [1.5 : 0.05 : 1.95]$. Be sure to note your cutoff point with the residual as your convergence criterion.

Using a cutoff of 10^{-6} as the maximum residual, I obtained the following:



5. Use Gauss's law to calculate the capacitance per unit length of your simulated capacitor. Compare against the common analytical solution using

$$C' = \frac{\epsilon_0 W}{d} \quad \text{(F/m)} \ ,$$

where W is the width of the plates, d is the separation distance, and ϵ_0 is the permittivity of free space.

- How well does your simulated value agree with the analytical computation?

  The calculated capacitance comes in around 44 pF. The simulation produces 65 pF, which is 46% greater.

- Which value do you think is the more accurate solution? The analytical or the simulated? Why?

  The simulation is almost certainly more accurate.

- What assumptions does the analytical expression make? (hint: think back to your previous courses in electromagnetics when you first derived this equation) What errors exist in the simulation model? How could you maybe remove those errors?

  The analytical expression makes several assumptions about the nature of the electric field that is not entirely accurate for this model. It assumes a uniform electric field intensity between the plates as well as zero electric field intensity outside of the plates. These assumptions are not accurate when the capacitor plates are separated so far apart as in the model. The simulation model accounts for the fringing effects near the edges of the plates, as well as any nonzero fields outside of the plates. This gives a much more accurate result. However, the simulation also has errors of its own. The discretization of the grid is a significant source, plus the proximity of the boundaries to the plates. However, these can be alleviated by using a smaller grid spacing and by placing the boundaries further away.