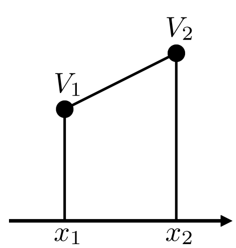
ECE5340/6340: Homework 12
Finite Element Method

Department of Electrical and Computer Engineering
University of Utah, Salt Lake City, Utah
April 4, 2012

Write your section (ECE5340 or ECE6340) by your name. Turn in a printed copy containing the problem solutions, plots, and the code used to generate them. Remember to comment and format the code so it is legible to the graders. Label the plots appropriately, including units for each axis and for the values plotted. Assume all units to be SI units unless stated differently. Due Wednesday 4/11 BEFORE class begins.

ASSIGNMENT

1. (10 points) Consider the one-dimensional element depicted below:



   Derive the elemental shape functions $\alpha_1(x)$ and $\alpha_2(x)$ for linear interpolation between the two samples.

2. (10 points) Derive the elemental stiffness matrix for an arbitrary element in 1D. Show that the total energy contained by the field $\mathbf{E}$ in the nth element can be expressed in a matrix-vector notation given by

$$W_n = \frac{1}{2} h\epsilon_0 |\mathbf{E}_n|^2 = \frac{1}{2}\epsilon_0\, \mathbf{v}_n^T \mathbf{C}_n \mathbf{v}_n \ .$$

3. (10 points) Now let us solve for the total energy in an arbitrary system of elements in 1D. Start by writing out the total energy as the summation of all the elemental energies:

$$W_{total} = \frac{1}{2}\epsilon_0 \left[\mathbf{v}_1^T \mathbf{C}_1 \mathbf{v}_1 + \mathbf{v}_2^T \mathbf{C}_2 \mathbf{v}_2 + \cdots + \mathbf{v}_N^T \mathbf{C}_N \mathbf{v}_N \right] \ .$$

   Show that we can rewrite this expression using a single matrix-vector equation with the form

$$W_{total} = \frac{1}{2}\epsilon_0\, \mathbf{v}^T \mathbf{C} \mathbf{v} \ .$$

   Write out the explicit expressions for the global unknown vector $\mathbf{v}$ and the global coefficient matrix $\mathbf{C}$.

4. (10 points) Consider the total energy in a four-node system. In matrix-vector notation, this will appear as an expression with the form

$$W = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

Suppose now that V_1 and V_4 are prescribed nodes. We therefore wish to rearrange the rows and columns in such a away as to segregate these nodes into distinct blocks within the matrix-vector expression. For example, if we swap V_1 with V_2, the above expression can likewise be written as

$$W = \begin{bmatrix} V_2 & V_1 & V_3 & V_4 \end{bmatrix} \begin{bmatrix} C_{22} & C_{21} & C_{23} & C_{24} \\ C_{12} & C_{11} & C_{13} & C_{14} \\ C_{32} & C_{31} & C_{33} & C_{34} \\ C_{42} & C_{41} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} V_2 \\ V_1 \\ V_3 \\ V_4 \end{bmatrix}$$

Notice how this simply swaps column 2 with column 1, followed by swapping row 2 with row 1.

Given this rule for row/column pivoting in matrix-vector equations, derive the global coefficient matrix that will result if we now swap V_1 with V_3. In other words, fill the empty matrix below:

$$W = \begin{bmatrix} V_2 & V_3 & V_1 & V_4 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_1 \\ V_4 \end{bmatrix}$$

5. (10 points) Assume that the $\mathbf{v}$ is a combination of free (variable) and prescribed (fixed) nodes, $\mathbf{v}_f$ and $\mathbf{v}_p$. This allows us to express $\mathbf{v}$ as a block vector with the form

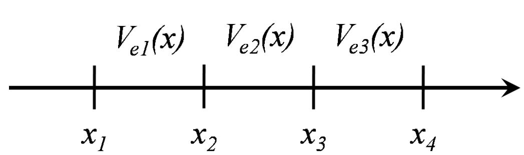
$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_f & \mathbf{v}_p \end{bmatrix} .$$

Given this, we can then redefine the global coefficient matrix in a similar manner. The total energy in the system is therefore written as

$$W_{total} = \frac{1}{2}\epsilon_0 \begin{bmatrix} \mathbf{v}_f & \mathbf{v}_p \end{bmatrix} \begin{bmatrix} \mathbf{C}_{ff} & \mathbf{C}_{fp} \\ \mathbf{C}_{pf} & \mathbf{C}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{v}_f \\ \mathbf{v}_p \end{bmatrix} .$$

Show that the minimum energy of the system produces a matrix-vector expression with the form $\mathbf{Ax} = \mathbf{b}$ (HINT: Take the derivative with respect to $\mathbf{v}_f$ and set it to zero).

6. (50 points) Let us now work out a simple example using the four nodes and three elements depicted below.



Assume that $V_1 = 1.0$ V and $V_4 = 0$ V (prescribed nodes), and that the node positions are given as

$$x_1 = 0 \text{ m}$$
$$x_2 = 1 \text{ m}$$
$$x_3 = 1.5 \text{ m}$$
$$x_4 = 2.25 \text{ m}$$

Part (A): Derive the elemental coefficient matrices for each of the three elements.

$$\mathbf{C}_1 =$$
$$\mathbf{C}_2 =$$
$$\mathbf{C}_3 =$$

Part (B): Derive the global coefficient matrix by zero-padding the elmental coefficient matrices. Show that the global coefficient matrix satisfies

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 \ .$$

Part (C): Rearrange the rows and columns within $\mathbf{v}$ and $\mathbf{C}$ so that the fixed nodes and prescribed nodes are segregated into distinct blocks. Show that the total energy W_{total} of the system is still the same by proving

$$\begin{aligned} W_{total} &= \frac{1}{2}\epsilon_0 \mathbf{v}^T \mathbf{C} \mathbf{v} \\ &= \frac{1}{2}\epsilon_0 \begin{bmatrix} \mathbf{v}_f & \mathbf{v}_p \end{bmatrix} \begin{bmatrix} \mathbf{C}_{ff} & \mathbf{C}_{fp} \\ \mathbf{C}_{pf} & \mathbf{C}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{v}_f \\ \mathbf{v}_p \end{bmatrix} \ . \end{aligned}$$

Write out the complete elements in the sub-matrices:

$$\mathbf{C}_{ff} =$$
$$\mathbf{C}_{fp} =$$
$$\mathbf{C}_{pf} =$$
$$\mathbf{C}_{pp} =$$

HINT: Use the following conventions:

$$\begin{aligned} \mathbf{v} &= \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix} \\ \mathbf{v}_f &= \begin{bmatrix} V_2 & V_3 \end{bmatrix} \\ \mathbf{v}_p &= \begin{bmatrix} V_1 & V_4 \end{bmatrix} \end{aligned}$$

Part (D): Solve for the two free node voltages that minimize the total energy of the system and plot the results. (HINT: You should get a straight line between V_1 and V_4).

EXTRA CREDIT (20 Points): Derive the elemental shape functions for linear interpolation between three nodes in two dimensions. Show that

$$V(x, y) = V_1\, \alpha_1(x, y) + V_2\, \alpha_2(x, y) + V_3\, \alpha_3(x, y) \ .$$