ECE5340/6340: Homework 11
ANSWER KEY

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University of Utah, Salt Lake City, Utah
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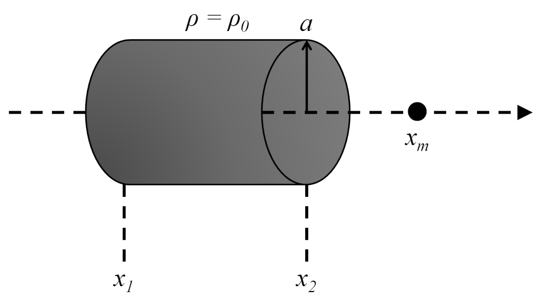
Write your section (ECE5340 or ECE6340) by your name. Turn in a printed copy containing the problem solutions, plots, and the code used to generate them. Remember to comment and format the code so it is legible to the graders. Label the plots appropriately, including units for each axis and for the values plotted. Assume all units to be SI units unless stated differently. Due Wednesday 4/4 BEFORE class begins.

ASSIGNMENT

1. (10 points) Calculate the voltage potential V at the point x_m along the axis of a hollow tube. Assume the tube has a uniform charge density ρ_0 spread out along the surface with a radius a. See the figure below:



HINT: Set the problem up as the following integral:

$$V(x_m) = \frac{a}{4\pi\epsilon_0} \int_{x_1}^{x_2} \int_{0}^{2\pi} \frac{\rho_0}{\sqrt{(x_m - x')^2 + a^2}} \, d\theta dx'$$

What is the voltage potential at the midpoint of the tube where $x_m = (x_1 + x_2)/2$?

The general expression for voltage potential is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV'$$

Now make the following substitutions:

$$\mathbf{r} = x_m \,\hat{\mathbf{x}}$$
$$\mathbf{r}' = x' \,\hat{\mathbf{x}} + y' \,\hat{\mathbf{y}} + z' \,\hat{\mathbf{z}}$$
$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y')^2 + (z')^2}$$

Convert y' and z' into cylindrical coordinates.

$$\begin{aligned} y' &= a\cos(\theta') \\ z' &= a\sin(\theta')\ . \end{aligned}$$

Note that we could equivalently interchange the sin and cosine functions, due to the radial symmetry of the problem. Given this, we now substitute

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + a^2}\ .$$

Next we note that the charge density in 2D becomes a surface charge density with units of C/m^2. We further note that the charge density is a constant value over the entire interval such that

$$\rho(x') = \rho_0\ , \qquad (\mathrm{C/m^2})\ .$$

Finally, the differential volume dV' term is determined by the differential surface area of a cylinder:

$$dV' = a\,d\theta' dx'$$

Putting it all together:

$$V(x_m) = \frac{a}{4\pi\epsilon_0}\int_{x_1}^{x_2}\int_{0}^{2\pi}\frac{\rho_0}{\sqrt{(x_m-x')^2+a^2}}\,d\theta' dx' \tag{1}$$

Now we insert a u-substitution.

$$\begin{aligned} u &= x_m - x' \\ du &= -dx' \\ u_1 &= x_m - x_1 \\ u_2 &= x_m - x_2 \end{aligned}$$

The integral can therefore simplify into

$$V(x_m) = -\frac{a}{2\epsilon_0}\int_{u_1}^{u_2}\frac{\rho_0}{\sqrt{u^2+a^2}}\,du$$

The solution is found either through trigonometric substitution, or (if you're lazy) through integration tables. Skipping ahead to the end, we have

$$\boxed{V(x_m) = \frac{a\rho_0}{2\epsilon_0}\ln\left|\frac{(x_m-x_1)+\sqrt{(x_m-x_1)^2+a^2}}{(x_m-x_2)+\sqrt{(x_m-x_2)^2+a^2}}\right|}\ .$$

The voltage at the midpoint of the tube is then found by plugging in $x_m = (x_1+x_2)/2$. The result is an expression similar to Equation (26) from the notes. If we define $h = (x_2 - x_1)$, the result is found to be

$$V(x_1/2 + x_2/2) = \frac{a\rho_0}{2\epsilon_0}\ln\left|\frac{h+\sqrt{h^2+4a^2}}{h-\sqrt{h^2+4a^2}}\right|\ .$$

2. (10 points) Consider a thin metal rod fixed at a constant voltage V_0 with some charge density $\rho(x)$ along its surface. If we break up the rod into N uniform segments, we may approximate the true distribution of charge using an expansion function with the form

$$\hat{\rho}(x') = \sum_{n=1}^{N} \alpha_n u_n(x') \ .$$

Using the Dirac delta function as the basis u_n, calculate the residual function $R(x)$ that results from this approximation.

This is just basically asking to derive Equation (19) from the notes. The result is

$$R(x) = V(x) - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} \frac{\alpha_n}{|x - x'_n|} \ .$$

3. (10 points) How many equations and how many unknowns does the residual R from the previous problem represent? Is this an overdetermined or an underdetermined system of equations? Describe how we might manipulate this problem in order to generate a *unique* solution for the unknown expansion coefficients in α_n.

At this point we have one equation with N unknown expansion coefficients, thus making an overdetermined system of equations. We solve this problem by *testing* the voltage at various points where the voltage potential is a known value (like the points along the rod) and then driving the residual to zero at each test location. If we therefore choose N test locations, we'll have N equations with N unknowns, and a thus a unique solution.

4. (10 points) Based on your answer to the previous problem, generate a system of linear equations that will uniquely determine the α_n coefficients. Explain your choice of *test* locations for x_m. Write out the general form for matrix coefficients $\mathbf{A}_{mn}$, including the self-terms.

Ideally, we want our test locations to be spread out within the space of the rod. This attempts to force a low residual at a wide distribution of samples along the rod. A bad choice would be a series of test locations that are clustered together. This forces the algorithm to push the residual to zero at only a small section of the rod. It also tends to produce an ill-conditioned matrix, which is likewise bad.

Following the derivation in the notes, we find ourselves with a system of linear equations with the form of

$$V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} \frac{\alpha_n}{|x_1 - x'_n|} = 0$$

$$V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} \frac{\alpha_n}{|x_2 - x'_n|} = 0$$

$$V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} \frac{\alpha_n}{|x_3 - x'_n|} = 0$$

$$\vdots$$

$$V_0 - \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} \frac{\alpha_n}{|x_M - x'_n|} = 0 \ .$$

We can express this as a matrix-vector equation $\mathbf{Ax} = \mathbf{b}$, where

$$A_{nm} = \frac{1}{x_m - x'_n} \ .$$

The diagonal elements in this matrix (the self-terms) are singular, so we fix this by applying the charged tube approximation along these values. These are simply Equation (27) from the notes:

$$\mathbf{A}_{mm} = \frac{1}{\Delta x} \ln \left| \frac{\Delta x + \sqrt{\Delta x^2 + 4a^2}}{\Delta x - \sqrt{\Delta x^2 + 4a^2}} \right| .$$

Finally, the vector **b** is just a set of constant values given by

$$b_n = 4\pi\epsilon_0 V_0 .$$

5. (10 points) Explain why a charged metal rod must necessarily rest at a fixed voltage potential in a static system. What is the electric field inside of a charged metal rod? What would the charges do if the voltage were not constant along the rod?

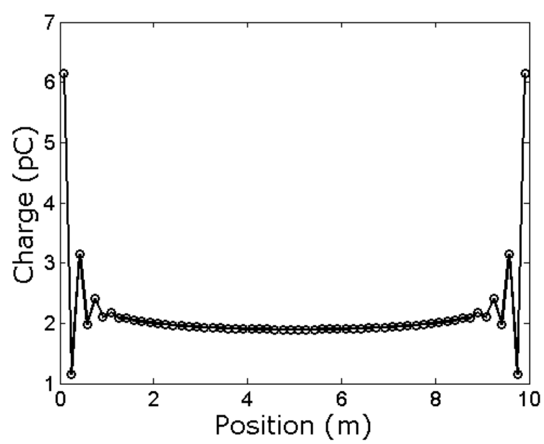
   The voltage potential along a perfect conductor is ALWAYS a constant value. Remember that this is equivalent to saying that the electric field inside of a perfect conductor is ALWAYS ZERO. If there were a nonzero field within the rod, then there would have to exist an electrical current and the system would no longer be static.

6. (50 points) Using the method of moments, write a Matlab function that calculates the estimation function $\hat{\rho}$ for the charge distribution along a thin metal wire held at constant potential. Use the following parameters as input arguments:

$$\begin{aligned} L &= \text{Length of the wire (m)} \\ a &= \text{Wire radius (m)} \\ V_0 &= \text{Wire potential (V)} \\ h &= \text{Length of the wire subdivisions (m)} \end{aligned}$$

   Assume the use of delta functions for your basis in $\hat{\rho}$. Demonstrate your code by plotting $\hat{\rho}$ for the case of $L = 10$ m, $a = L/100$, $V_0 = 1.0$ V, and $h = L/60$. Calculate the total charge on the rod and comment on any peculiar behavior you notice.

   The solution for the charge distribution is shown below:



   Notice how the charges are trying to push themselves away from the center and toward the edges. Also notice how there is some apparent instability around the edges. This is due to our assumption of point charges along the rod. At very coarse values for h and small values for a, this assumption can work fairly well and the solutions tend to look a little more reasonable. But as h gets small, the algorithm starts blowing up. If we truly wanted a stable solution, we would have to model the distribution of charges more accurately.

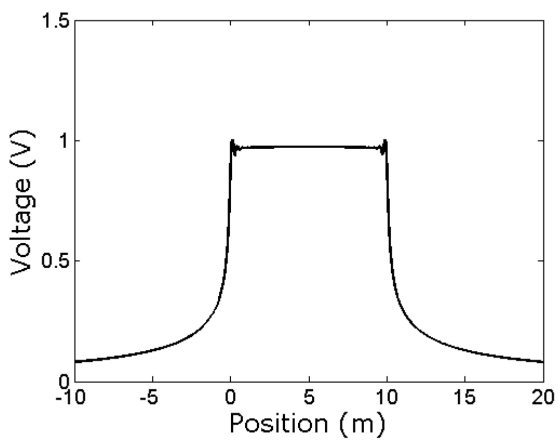
If done correctly, the total charge along the rod should be roughly 130 pC. Since this part can get pretty tricky, anything that produces a crude approximation to the above solution is probably good enough for beginners.

7. ECE 6340 Only: (10 points) Plot $V(x)$ along $x \in [-L, 2L]$, where the rod sits along $[0, L]$. Assume your point charges in $\hat{\rho}$ are distributed uniformly along hollow tube segments with length h as shown in Problem 1. Comment on any observations you make.

   Remember to distribute each charge along the tube surface so that

$$\rho_n = \frac{\alpha_n}{2\pi a h} \ .$$

   The voltage plot should then look like



   Note how it is reasonably flat along the interior of the rod, just like we expected. There is also some instability around the edges of the rod, due to the fact that we have not accounted for any charges along the end caps.

8. EXTRA CREDIT: (20 points) Repeat the MoM problem using rectangle functions as the basis for $\hat{\rho}$. In other words, assume that the entire rod is a hollow cylindrical tube rather than point charges, and that the charges are uniformly distributed along little segments with length h. Plot $V(x)$ along $x \in [-L, 2L]$, where the rod sits along $[0, L]$. Comment on any observations you make. How could we better approximate a true metal rod?