Chapter-5 (contd)

Lossy Media

For normal incidence in a lossy media we will add an additional term $e^{-\alpha_1 z}$ & $e^{-\alpha_2 z}$ to the incident, reflected and transmitted electric & magnetic fields in medium 1 & 2 respectively.

for eg:-
$$\bar{E}^i = \hat{a}_x E_0 e^{-\alpha_1 z} e^{-j\beta_1 z}$$
$$\bar{H}^i = \hat{a}_y \frac{E_0}{\eta_1} e^{-\alpha_1 z} e^{-j\beta_1 z}$$

The ~~con~~ attenuation constant α_i, phase constant β_i and intrinsic impedance η_i are related to ε_i, μ_i & σ_i by the following relationship.

	Exact	Good dielectric	Good conductor
α	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{1/2}$	$\simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
β	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]\right\}^{1/2}$	$\simeq \omega\sqrt{\mu\varepsilon}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
η	$= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\simeq \sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$

Total Electric and magnetic fields in medium 1

$$\bar{E}^1 = \bar{E}^i + \bar{E}^r = \hat{a}_x E_o e^{-\alpha_1 z} e^{-j\beta_1 z} \left(1 + \Gamma^b e^{2\alpha_1 z} e^{j2\beta_1 z}\right)$$

$$\bar{H}^1 = \bar{H}^i + \bar{H}^r = \hat{a}_y \frac{E_o}{\eta_1} e^{-\alpha_1 z} e^{-j\beta_1 z} \left(1 - \Gamma^b e^{2\alpha_1 z} e^{j2\beta_1 z}\right)$$

OBLIQUE INCIDENCE: Dielectric Conductor Interface

Assume a uniform plane wave is obliquely incident upon a planar interface where medium 1 is a perfect dielectric and medium 2 is lossy. The transmitted wave can be written as:

$$\bar{E}_\perp^t = \hat{a}_y E_\perp^t e^{-j\beta^t \cdot r} = \hat{a}_y T_\perp^b E_o e^{-j\beta_2 (x \sin\theta_t + z \cos\theta_t)}$$

or

$$\bar{E}_\parallel^t = (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t)\, T_\parallel^b E_o e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)}$$

In general for lossy medium we can write

$$\bar{E}^t = \bar{E}_2 \exp\left[-\gamma_2 (x \sin\theta_t + z \cos\theta_t)\right] \qquad (1)$$

For lossy media Snell's Law can be written as

$$\gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

$$\sin\theta_t = \frac{\gamma_1}{\gamma_2} \sin\theta_i = \frac{j\beta_1}{\alpha_2 + j\beta_2} \sin\theta_i$$

& $\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{j\beta_1}{\alpha_2+j\beta_2}\right)^2 \sin^2\theta_i} = s e^{j\xi}$

$$= s\,(\cos\xi + j\sin\xi)$$

Substituting in (1)

$$\bar{E}^t = \bar{E}_2 \exp\left\{-(\alpha_2+j\beta_2)\left[x\,\frac{j\beta_1}{\alpha_2+j\beta_2}\sin\theta_i + z\,s\,(\cos\xi + j\sin\xi)\right]\right\}$$

which reduces to

$$\bar{E}^t = \bar{E}_2 \exp\left[-zs\,(\alpha_2\cos\xi - \beta_2\sin\xi)\right]$$
$$\times \exp\left\{-j\left[\beta_1 x\sin\theta_i + zs\,(\alpha_2\sin\xi + \beta_2\cos\xi)\right]\right\}$$

$$\bar{E}^t = \bar{E}_2\, e^{-zp} \exp\left[-j(\beta_1 x\sin\theta_i + zq)\right] \qquad \text{— (2)}$$

where

$$\left.\begin{aligned} p &= s\,(\alpha_2\cos\xi - \beta_2\sin\xi) = \alpha_{2e} \\ q &= s\,(\alpha_2\sin\xi + \beta_2\cos\xi) \end{aligned}\right\} \qquad \text{— (3)}$$

The instantaneous electric field can be written as:-

$$\bar{\mathcal{E}}^t = Re\left(\bar{E}^t e^{j\omega t}\right) = \bar{E}_2\, e^{-zp}\, Re\left(\exp\left\{j\left[\omega t - (\beta_1 x\sin\theta_i + zq)\right]\right\}\right)$$

$$\bar{\mathcal{E}}^t = \bar{E}_2\, e^{-zp} \cos\left[\omega t - (\beta_1 x\sin\theta_i + zq)\right] \qquad \text{— (4)}$$

The constant amplitude planes are parallel to the interface and the constant phase planes are inclined at an angle ψ_2 that is no longer θ_t

We can write

$$\omega t - (\beta_1 x \sin\theta_i + zq) = \omega t - \sqrt{(\beta_1 \sin\theta_i)^2 + q^2} \times \left[\frac{(\beta_1 \sin\theta_i)\, x}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} + \frac{q z}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} \right] \quad (5)$$

If we define an angle φ_2 such that

$$u = \beta_1 \sin\theta_i$$

$$\sin\varphi_2 = \frac{\beta_1 \sin\theta_i}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} = \frac{u}{\sqrt{u^2 + q^2}} \quad - (6)$$

$$\cos\varphi_2 = \frac{q}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} = \frac{q}{\sqrt{u^2 + q^2}} \quad - (7)$$

or we can write

$$\varphi_2 = \tan^{-1}\left(\frac{\beta_1 \sin\theta_i}{q}\right) = \tan^{-1}\left(\frac{u}{q}\right) \quad - (8)$$

We can write (2) as:

$$\bar{\mathcal{E}}^t = \bar{E}_z e^{-zp} \,\mathrm{Re}\left(\exp\left\{j\left[\omega t - \sqrt{u^2+q^2}\left(\frac{ux}{\sqrt{u^2+q^2}} + \frac{qz}{\sqrt{u^2+q^2}}\right)\right]\right\}\right)$$

$$= \bar{E}_z e^{-zp} \,\mathrm{Re}\left[\exp\{j[\omega t - \beta_{2e}(x\sin\varphi_2 + z\cos\varphi_2)]\}\right]$$

$$\bar{\mathcal{E}}^t = \bar{E}_z e^{-zp} \,\mathrm{Re}\left[\exp\{j[\omega t - \beta_{2e}(\hat{n}_\varphi \cdot \bar{r})]\}\right) \quad - (9)$$

where

$$\left.\begin{aligned} \hat{n}_\psi &= \hat{a}_x \sin\psi_2 + \hat{a}_z \cos\psi_2 \\ \beta_{2e} &= \sqrt{u^2+q^2} \end{aligned}\right\} \quad - (10)$$

The true angle of refraction is ψ_2 and not θ_2

The wave travels along a direction defined by unit vector $\hat{n}_\psi$

The constant phase planes are $\perp^r$ to unit vector $\hat{n}_\psi$.

The phase velocity of the wave in medium 2 is obtained by setting the exponent of (9) to a constant and differentiating it with respect to time. We get,

$$\omega(1) - \sqrt{u^2+q^2}\left(\hat{n}_\psi \cdot \frac{d\bar{r}}{dt}\right) = 0$$

$$\omega(1) - \sqrt{u^2+q^2}\left(\hat{n}_\psi \cdot \frac{d\bar{r}}{dt}\right) = \omega - \beta_{2e}(\hat{n}_\psi \cdot v_p) = 0$$

or

$$v_{pr} = \frac{\omega}{\beta_{2e}} = \frac{\omega}{\sqrt{u^2+q^2}} = \frac{\omega}{(\beta_1 \sin\theta_i)^2 + q^2}$$

OBLIQUE INCIDENCE :- CONDUCTOR - CONDUCTOR INTERFACE

Read T.B pg 214 – 220

Normal Incidence on a lossy Medium

In practice, all media have some loss, leading to absorption of the transmitted energy as it propagates through the lossy medium. In some cases such attenuation may be undesirable but unavoidable; in other cases the heat produced by the attenuation of the wave in the lossy material may constitute the primary application. Here we talk about two important cases involving reflection from an imperfect conducting plane and multiple lossy interfaces.

I. Reflection from an Imperfect Conducting Plane

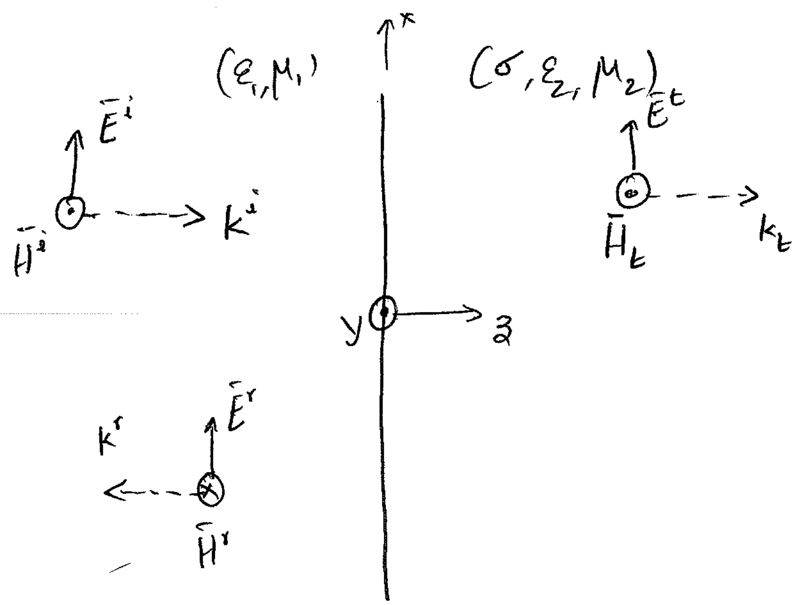
We consider a special case of incidence of a uniform plane wave on a "good" conductor with finite conductivity σ. We will show that the total current flowing within the conducting material is essentially independent of the conductivity. As the conductivity approaches infinity, the total current is squeezed into a narrower and narrower layer, until in the limit a true surface current is obtained.

- We will further show that the conductor can be characterized as a boundary exhibiting a surface impedance of

$$Z_s = \frac{1+j}{\sigma\delta} \qquad \delta \Rightarrow \text{skin depth for the conductor}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

Let a uniform plane wave be normally incident on a conducting interface at $z=0$ as shown in figure below



The phasor fields for the incident reflected and transmitted wave is given as.

$$\bar{E}^i(z) = \hat{x} E_o e^{-j\beta_1 z}$$

$$\bar{H}^i(z) = \hat{y} \frac{E_o}{\eta_1} e^{-j\beta_1 z}$$

$$\bar{E}^r(z) = \hat{x} \Gamma E_o e^{j\beta_1 z}$$

$$\bar{H}^r(z) = -\hat{y} \frac{\Gamma E_o}{\eta_1} e^{j\beta_1 z}$$

$$\bar{E}^t(z) = \hat{x} T E_o e^{-\gamma_2 z}$$

$$\bar{H}^t(z) = \hat{y} \frac{T E_o}{\eta_2} e^{-\gamma_2 z}$$

$$\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1} \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \simeq 377\,\Omega$$

$$\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)} \qquad \eta_2 = \sqrt{\frac{j\omega\mu_2}{(\sigma_2 + j\omega\varepsilon_2)}}$$

In a good conducting medium ($\sigma >> \omega\varepsilon$) we have

$\nabla \times \bar{H} = (j\omega\varepsilon + \sigma)\,\bar{E} \simeq \sigma\bar{E}$ since the conduction current is greater than displacement current

we can rewrite

$$\nabla \times \bar{H} = j\omega\left[\frac{\sigma}{(j\omega)}\right]\bar{E}$$

$\varepsilon_{eff} = \frac{\sigma}{j\omega}$ may be considered as the permittivity ε in Maxwell's equations in a lossless medium. Thus, propagation constant and intrinsic impedance of the lossy medium can be obtained from that of a lossless medium by the substitution $\varepsilon_{eff} \rightarrow \varepsilon$. We thus have a propagation constant for medium 2 by:

$$\gamma_2 = j\omega\sqrt{\mu\,\varepsilon_{eff}} \simeq j\omega\left(\frac{\mu_2\sigma_2}{j\omega}\right)^{1/2} = (j\omega\mu_2\sigma_2)^{1/2}$$

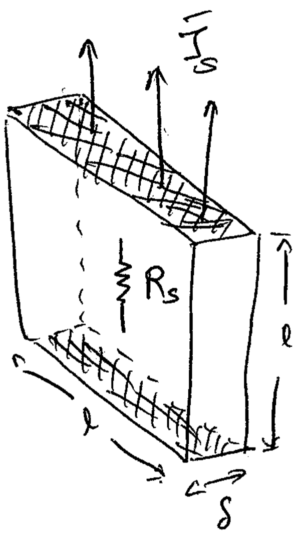
$$= \delta^{-1}(1+j)$$

and the complex intrinsic impedance is given by:-

$$\eta_c = Z_s = R_s + jX_s = \sqrt{\frac{\mu_2}{\varepsilon_{eff}}} \simeq \left(\frac{j\omega\mu_2}{\sigma_2}\right)^{1/2} = \frac{\gamma_2}{\sigma_2} = (\sigma_2\delta)^{-1}(1+j)$$

$\delta = (\pi f \mu_2 \sigma_2)^{-1/2}$ is skin depth for medium 2.

The conductor presents an impedance $Z_s = \eta_c$ to the electromagnetic wave with equal inductive and resistive part, defined above as R_s and jX_s, respectively. The resistance part is simply the resistance of a sheet of metal of $1\,m^2$ and of thickness δ as shown in figure below



The resistance between two shaded faces (perpendicular to the current flow) is given by

$$R_s = \frac{l}{l\delta\sigma_2} = \frac{1}{\delta\sigma_2}$$

Since the resistance is independent of the linear dimension, l, it is called a surface resistance and the complex intrinsic impedance η_c can be thought of a a surface impedance of the conductor

The reflection and transmission coefficients can be written as.

$$\Gamma = \frac{Z_s - \eta_1}{Z_s + \eta_1} = \rho e^{j\phi_1} \qquad T = \frac{2Z_s}{Z_s + \eta_1} = \tau e^{j\phi_\tau}$$

For a reasonably good conductor, Z_s is very small compared to η_1 for free space. For example for copper ($\sigma_2 = 5.8 \times 10^7$ S/m). At 1MHz $\delta \cong 66.1\,\mu m$ and $R_s = 2.61 \times 10^{-8}\,\Omega$

The reflection and Transmission coefficients are given as $\Gamma \cong 0.999986\, e^{j179.99992 1^\circ}$ and $T \cong 2 \times 10^{-6} e^{j45^\circ}$. We know that for all practical purposes the field in front of the conductor ($z<0$) is the same as exists for a perfect conductor, since $\rho = |\Gamma|$ is very close to unity. For the same reason, only a very small amount of power is transmitted into the conductor. Nevertheless, this small amount of power that penetrates into the wall can cause significant attenuation of waves propagating in waveguides & coaxial lines especially when these types of transmission lines are relatively long.

If medium 2 is a good conductor $(\sigma_2 >> \omega \varepsilon_2)$

we get

$$\Gamma = \frac{\delta - \lambda_1 (1-j)}{\delta + \lambda_1 (1-j)} \quad \text{and} \quad \tau = \frac{2\delta}{\lambda_1 (1-j)}$$

$\lambda_1 = \frac{2\pi}{\beta_1}$ is wavelength of first medium.

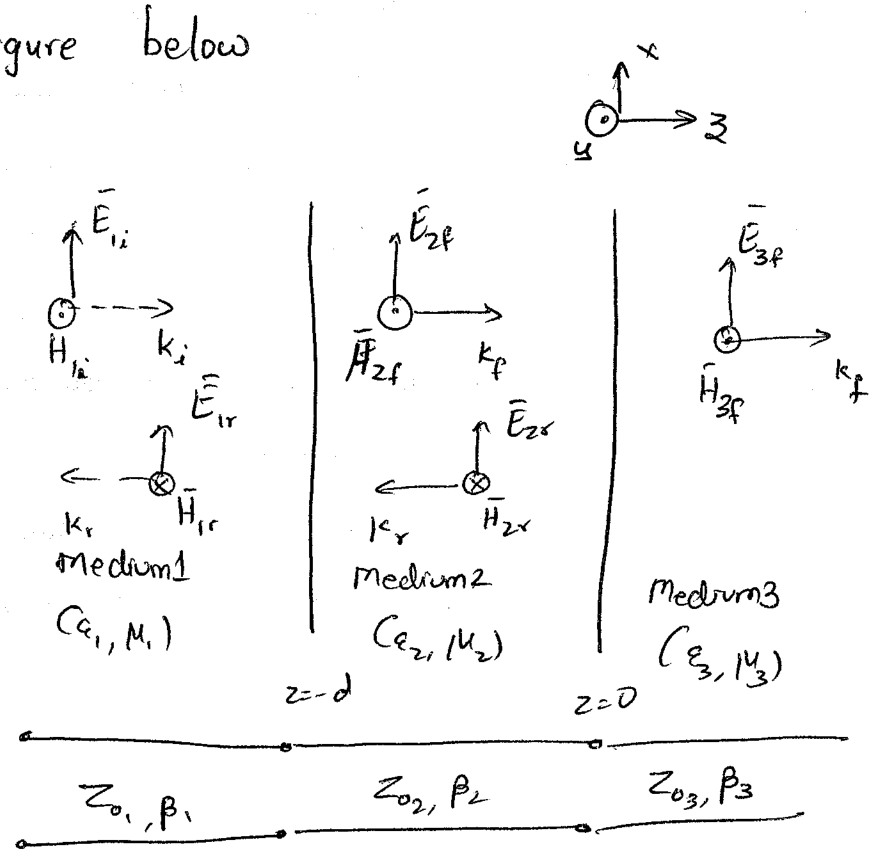
Since $\lambda_1 >> \delta$ we have $\Gamma \simeq e^{j\pi}$ $\tau = |\tau| << 1$

Multiple Dielectric Interfaces

Many practical applications involves the reflection and refraction of electromagnetic waves from dielectric or metallic surfaces that are coated with another dielectric material to reduce reflections and to improve the coupling of the wave energy.

Some of the practical application of reflection from multiple dielectric interfaces include antireflective coatings to improve light transmission of a lens, thin film coating on optical components to reduce losses selectively over narrow wavelength ranges, metal mirror coating, flexible metal coils for EMI shielding etc.

A simple ~~die~~ multilayer problem can be formulated as shown in the figure below



η_1, η_2 and η_3 are intrinsic impedance of the three medium and are separated by infinite planar boundary. at $z=-d$ and $z=0$

The total Electric and magnetic field in the three media are given as,

$$\bar{E}_1(z) = \bar{E}_{1i} + \bar{E}_{1r} = \hat{x} E_0 \left[e^{-j\beta_1(z+d)} + \Gamma_{eff}\, e^{j\beta_1(z+d)} \right]$$

$z < -d$

$$\bar{H}_1(z) = \bar{H}_{1i} + \bar{H}_{1r} = \hat{y} \frac{E_0}{\eta_1} \left[e^{-j\beta_1(z+d)} - \Gamma_{eff}\, e^{j\beta_1(z+d)} \right]$$

$$\bar{E}_2(z) = \bar{E}_{2f} + \bar{E}_{2r} = \hat{x} E_{20} \left[e^{-j\beta_2 z} + \Gamma_{23}\, e^{j\beta_2 z} \right]$$

$$\bar{H}_2(z) = \bar{H}_{2f} + \bar{H}_{2r} = \hat{y} \frac{E_{20}}{\eta_2} \left[e^{-j\beta_2 z} - \Gamma_{23}\, e^{j\beta_2 z} \right]$$

$$\bar{E}_3(z) = \bar{E}_{3f} = \hat{x}\, T_{eff}\, E_0\, e^{-j\beta_3 z}$$

$$\bar{H}_3(z) = \bar{H}_{3f} = \hat{y} \frac{T_{eff}\, E_0}{\eta_3} e^{-j\beta_3 z}$$

Γ_{23} is reflection coefficient at $z = 0$

Γ_{eff} is effective reflection coefficient at $z = -d$

$\mathcal{T}_{eff}$ is an effective transmission coefficient.

Γ_{eff} is complex. This can be understood using the transmission line analogy. The dielectric layer of thickness d simply acts as an impedance transformer, which takes the intrinsic impedance η_3 of medium 3 and presents it at $z=-d$ interface as another impedance which can be complex.

Applying Boundary conditions at $z=-d$

$$E_0(1+\Gamma_{eff}) = E_{20}\left(e^{j\beta_2 d} + \Gamma_{23}e^{-j\beta_2 d}\right)$$

$$\frac{E_0}{\eta_1}(1-\Gamma_{eff}) = \frac{E_{20}}{\eta_2}\left(e^{j\beta_2 d} - \Gamma_{23}e^{-j\beta_2 d}\right)$$

At $z=0$ we have

$$E_{20}(1+\Gamma_{23}) = \mathcal{T}_{eff}E_0$$

$$\frac{E_{20}}{\eta_2}(1-\Gamma_{23}) = \frac{\mathcal{T}_{eff}E_0}{\eta_3}$$

So we can find

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

$$\Gamma_{eff} = \frac{(\eta_2 - \eta_1)(\eta_3 + \eta_2) + (\eta_2 + \eta_1)(\eta_3 - \eta_2)e^{-2j\beta_2 d}}{(\eta_2 + \eta_1)(\eta_3 + \eta_2) + (\eta_2 - \eta_1)(\eta_3 - \eta_2)e^{-2j\beta_2 d}}$$

$$T_{eff} = \frac{4\eta_2 \eta_3 \, e^{-j\beta_2 d}}{(\eta_2 + \eta_1)(\eta_3 + \eta_2) + (\eta_2 - \eta_1)(\eta_3 - \eta_2)e^{-j2\beta_2 d}}$$

The same expressions can be applied for lossy case by replacing $j\beta_2$ by $\gamma_2 = \alpha_2 + j\beta_2$

The net time average power propagating in the z direction in medium 1 must be equal to that carried in the same direction by the transmitted wave in medium 3, because medium 2 is lossless and therefore cannot dissipate power.

Impedance Transformation and Transmission Line Analogy

The impedance at any position 'z' is given by:

$$Z(z) = \frac{[E_x(z)]_{total}}{[H_y(z)]_{total}}$$

$$E_{2x}(z) = E_{20}\left[e^{-j\beta_2 z} + \Gamma_{23} e^{j\beta_2 z}\right]$$

$$H_{2y}(z) = \frac{E_{20}}{\eta_2}\left[e^{-j\beta_2 z} - \Gamma_{23} e^{j\beta_2 z}\right]$$

which is analogous to

$$V(z) = V_{20}\left[e^{-j\beta_2 z} + \Gamma_L e^{j\beta_2 z}\right]$$

$$I(z) = \frac{V_{20}}{Z_{02}}\left[e^{-j\beta_2 z} - \Gamma_L e^{j\beta_2 z}\right]$$

V_{20} is constant $\quad \Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

For a 3 media problem, the wave impedance of the total field in medium 3 is

$$Z_3(z) = \frac{E_{3x}(z)}{H_{3x}(z)} = \eta_3$$

Since there is only one wave travelling in the medium three $Z_3(z) = \eta_3$ is independent of position z. This is similar to the line impedance of an infinitely long or match terminated transmission line.

In medium 2 the wave impedance has to take into account medium 3, which lies beyond the $z=0$ interface and acts as a load on the second transmission line. To find ratio of E_{2x} and H_{2y} we need to know the relative electric field amplitudes of the two traveling waves within the layer that is Γ_{23}.

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

$$Z_2(z) = \frac{E_{2x}(z)}{H_{2y}(z)} = \eta_2 \frac{e^{-j\beta_2 z} + \Gamma_{23} e^{j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_{23} e^{j\beta_2 z}}$$

At $z=0$ this expression reduces to

$$Z_2(z=0) = \eta_2 \frac{1+\Gamma_{23}}{1-\Gamma_{23}} = \eta_2 \frac{1+\frac{\eta_3-\eta_2}{\eta_3+\eta_2}}{1-\frac{\eta_3-\eta_2}{\eta_3+\eta_2}} = \eta_2 \frac{2\eta_3}{2\eta_2} = \eta_3$$

$$Z_2(-d) = \eta_2 \frac{e^{j\beta_2 d} + \Gamma_{23} e^{-j\beta_2 d}}{e^{j\beta_2 d} - \Gamma_{23} e^{-j\beta_2 d}} = \eta_2 \frac{\eta_3 + j\eta_2 \tan(\beta_2 d)}{\eta_2 + j\eta_3 \tan(\beta_2 d)}$$

$$= \eta_{23}$$

Using expression for total electric and magnetic field in region 1 we can write

$$Z_1(z) = \frac{E_{1x}(z)}{H_{1y}(z)} = \eta_1 \frac{e^{-j\beta_1(z+d)} + \Gamma_{eff} e^{j\beta_1(z+d)}}{e^{-j\beta_1(z+d)} - \Gamma_{eff} e^{j\beta_1(z+d)}}$$

At $z = -d$, this becomes

$$Z_1(z = -d) = \eta_1 \frac{1 + \Gamma_{eff}}{1 - \Gamma_{eff}}$$

Since EM boundary conditions require the continuity of the tangential electric (E_x) and magnetic (H_y) field components, and since wave impedance is defined as the ratio of these two quantities, the wave impedance on both sides of any interference must be equal.

$$Z_1(z = -d) = \eta_1 \frac{1 + \Gamma_{eff}}{1 - \Gamma_{eff}} = Z_2(z = -d) \qquad \Gamma_{eff} = \frac{Z_2(-d) - \eta_1}{Z_2(-d) + \eta_1}$$

$$= \frac{\eta_{23} - \eta_1}{\eta_{23} + \eta_1}$$

Reflection and Transmission of Multiple Interfaces

Reflection coefficient of a Single Slab Layer

For a normal incidence the reflection coefficient Γ^b at the boundary of a single planar interface is given by

$$\Gamma^b = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

At a distance $z = -l$ from the boundary is given by

$$\Gamma_{in}(z = -l) = \Gamma^b e^{-j2\beta_1 l}$$

$$\left.\frac{E^r(z)}{E^i(z)}\right|_{z=-l} = \left.\frac{\Gamma^b E_o e^{j\beta_1 z}}{E_o e^{-j\beta_1 z}}\right|_{z=-l} = \Gamma^b e^{-j2\beta_1 l_1}$$

Just to the right of the boundary the input impedance in the $+z$ direction is equal to intrinsic impedance η_2 of medium 2, that is.

$$Z_{in}(z = 0^+) = \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

The input impedance $Z_{in}(z=-\ell)$ is equal to

$$Z_{in}\Big|_{z=-\ell} = \frac{\vec{E}^{total}\Big|_{z=-\ell}}{\vec{H}^{total}\Big|_{z=-\ell}}$$

where

$$\bar{E}^{total}\Big|_{z=-\ell} = (\bar{E}^i+\bar{E}^r)\Big|_{z=-\ell} = E_o e^{j\beta_1 \ell}\left(1+\Gamma^b e^{-j2\beta_1 \ell}\right)$$

$$= E_o e^{j\beta_1 \ell}\left[1+\Gamma_{in}(\ell)\right]$$

$$\bar{H}^{total}\Big|_{z=-\ell} = (\bar{H}^i+\bar{H}^r)\Big|_{z=-\ell} = \frac{E_o}{\eta_1} e^{j\beta_1 \ell}\left(1-\Gamma^b e^{-j2\beta_1 \ell}\right)$$

$$= \frac{E_o}{\eta_1} e^{j\beta_1 \ell}\left(1-\Gamma_{in}(\ell)\right)$$

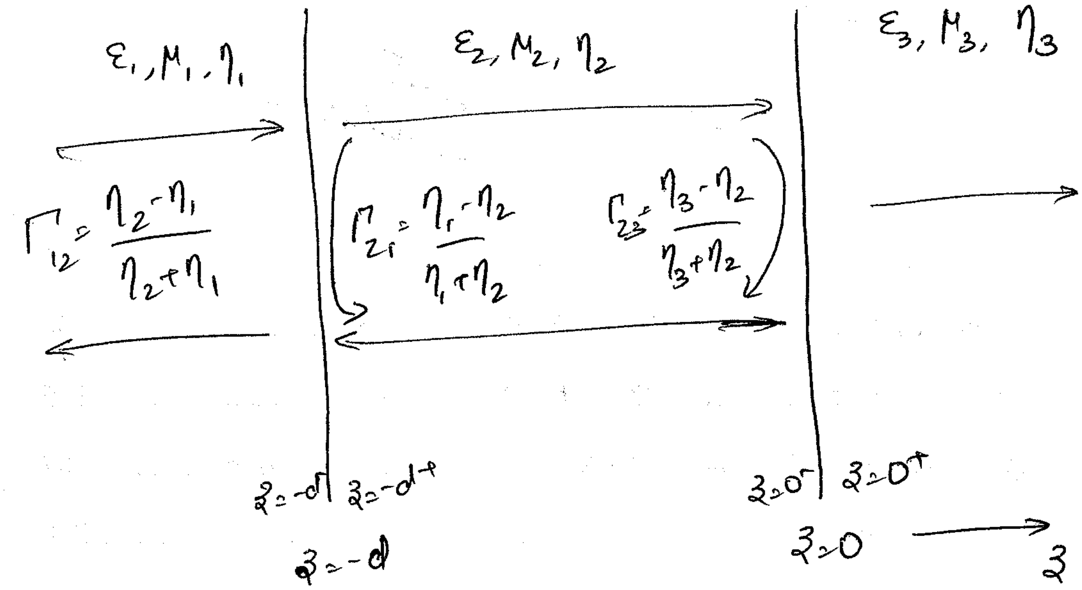
Therefore

$$Z_{in}\Big|_{z=-\ell} = \eta_1\left(\frac{1+\Gamma^b e^{-j2\beta_1 \ell}}{1-\Gamma^b e^{-j2\beta_1 \ell}}\right)$$

$$= \eta_1\left(\frac{\eta_2 + j\eta_1 \tan(\beta_1 \ell)}{\eta_1 + j\eta_2 \tan(\beta_1 \ell)}\right)$$

This is analogous to the well-known impedance transfer equation that is widely used in transmission line theory

Multilayer Analysis



At $z = -d$ the input impedance can be written as

$$Z_{in}(z=-d) = \eta_2 \left(\frac{1 + \Gamma_{in}(z=0^-)\, e^{-j2\beta_2 d}}{1 - \Gamma_{in}(z=0^-)\, e^{-j2\beta_2 d}} \right) = \eta_2 \left(\frac{(\eta_3+\eta_2) + (\eta_3-\eta_2)e^{-j2\beta_2 d}}{(\eta_3+\eta_2) - (\eta_3-\eta_2)e^{-j2\beta_2 d}} \right)$$

and the input reflection coefficient at $z = -d^-$ can be expressed as

$$\Gamma_{in}(z=-d^-) = \frac{Z_{in}(z=-d^+) - \eta_1}{Z_{in}(z=-d^+) + \eta_1}$$

$$= \frac{\eta_2\left[(\eta_3+\eta_2) + (\eta_3-\eta_2)e^{-j2\beta_2 d}\right] - \eta_1\left[(\eta_3+\eta_2) - (\eta_3-\eta_2)e^{-j2\beta_2 d}\right]}{\eta_2\left[(\eta_3+\eta_2) + (\eta_3-\eta_2)e^{-j2\beta_2 d}\right] + \eta_1\left[(\eta_3+\eta_2) - (\eta_3-\eta_2)e^{-j2\beta_2 d}\right]}$$

Using the intrinsic reflection coefficients as shown in the figure 5-17 the input reflection coefficients can be written as:

$$\Gamma_{in}(z=-d^-) = \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_2 d}}$$

Γ_{12} is the intrinsic reflection coefficient of the initial reflection and $T_{12}\Gamma_{23}T_{21}e^{-j2\theta}$, etc are the contributions to the input reflection that are due to multiple bounces within the medium 2 slab. The total input reflection coefficient can be written in a geometric series that takes the form of

$$\Gamma_{in}(z=-d^-) = \Gamma_{12} + \frac{T_{12}T_{21}\Gamma_{23}e^{-j2\beta_2 d}}{1+\Gamma_{21}\Gamma_{23}e^{-j2\beta_2 d}}$$

$$\Gamma_{21} = -\Gamma_{12}$$

$$T_{12} = 1 + \Gamma_{21}$$

$$T_{21} = 1 + \Gamma_{12}$$

Then

$$\Gamma_{in}(z=-d^-) = \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}{1+\Gamma_{12}\Gamma_{23}e^{-j2\beta_2 d}} \underset{|\Gamma_{23}| \ll 1}{\overset{|\Gamma_{12}| \ll 1}{\simeq}} \Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}$$