Construction of Solutions

The most widely used modes of propagation are the Transverse Electromagnetic, Transverse Electric, and Transverse magnetic.

Transverse Electromagnetic Modes: Source free region.

- Simplest forms of field configurations.
- referred to as the lowest-order modes.
- Both electric and magnetic field components are transverse to a given direction.

Rectangular coordinate system

$$\bar{E} = \bar{E}_A + \bar{E}_F = -j\omega \bar{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{A}) - \frac{1}{\varepsilon}\nabla\times\bar{F} \quad -(1)$$

Assuming the vector potentials $\bar{A}$ and $\bar{F}$ have solutions of the form

$$\bar{A}(x,y,z) = \hat{a}_x A_x(x,y,z) + \hat{a}_y A_y(x,y,z) + \hat{a}_z A_z(x,y,z) \quad -(2)$$

which satisfies with $\bar{J}=0$

$$\nabla^2\bar{A} + \beta^2\bar{A} = 0 \quad -(3)$$

or

$$\left.\begin{aligned} \nabla^2 A_x + \beta^2 A_x &= 0 \\ \nabla^2 A_y + \beta^2 A_y &= 0 \\ \nabla^2 A_z + \beta^2 A_z &= 0 \end{aligned}\right\} \quad -(4)$$

$|||^{ly}$

$$\bar{F}(x,y,z) = \hat{a}_x F_x(x,y,z) + \hat{a}_y F_y(x,y,z) + \hat{a}_z F_z(x,y,z) \quad -(5)$$

which with $\bar{M}=0$ satisfies

$$\nabla^2\bar{F}+\beta^2\bar{F}=0 \qquad -(6)$$

or

$$\left.\begin{aligned}\nabla^2F_x+\beta^2F_x=0\\ \nabla^2F_y+\beta^2F_y=0\\ \nabla^2F_z+\beta^2F_z=0\end{aligned}\right\} \qquad -(7)$$

Expanding (1) we get

$$\bar{E}=\hat{a}_x\left[-j\omega A_x-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2A_x}{\partial x^2}+\frac{\partial^2A_y}{\partial x\partial y}+\frac{\partial^2A_z}{\partial x\partial z}\right)-\frac{1}{\varepsilon}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\right]$$

$$+\hat{a}_y[\;---\;]+\hat{a}_z[\;---\;] \qquad -(8)$$

$|||^{ly}$.

$$\bar{H}=\bar{H}_A+\bar{H}_F=\frac{1}{\mu}\nabla\times\bar{A}-j\omega\bar{F}-j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{F}) \qquad -(9)$$

when expanded it becomes

$$\bar{H}=\hat{a}_x\left[-j\omega F_x-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2F_x}{\partial x^2}+\frac{\partial^2F_y}{\partial x\partial y}+\frac{\partial^2F_z}{\partial x\partial z}\right)+\frac{1}{\mu}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\right]$$

$$+\hat{a}_y\left[-j\omega F_y-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2F_x}{\partial x\partial y}+\frac{\partial^2F_y}{\partial y^2}+\frac{\partial^2F_z}{\partial y\partial z}\right)+\frac{1}{\mu}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right]$$

$$+\hat{a}_z\left[-j\omega F_z-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2F_x}{\partial x\partial z}\right.-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2F_x}{\partial x\partial z}+\frac{\partial^2F_y}{\partial y\partial z}+\frac{\partial^2F_z}{\partial z^2}\right)\right.$$

$$\left.+\frac{1}{\mu}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right] \qquad — \quad (10)$$

Derive expressions for $\bar{E}$ & $\bar{H}$ in terms of components of $\bar{A}$ and $\bar{F}$ potentials, that are TEM to the z direction (TEM^z)

→ Since we consider TEM waves in the 'z' direction we can assume $E_z = H_z = 0$

1) Assume

$$A_x = A_y = F_x = F_y = 0 \quad A_z \neq 0 \quad F_z \neq 0 \quad \frac{\partial}{\partial x} \neq 0 \quad \frac{\partial}{\partial y} \neq 0$$

From equation 6-41 we have.

$$\bar{E} = \hat{a}_x\left[-j\omega \bar{A}_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 \bar{A}_x}{\partial x^2} + \frac{\partial^2 \bar{A}_y}{\partial x \partial y} + \frac{\partial^2 \bar{A}_z}{\partial x \partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial \bar{F}_z}{\partial y} - \frac{\partial \bar{F}_y}{\partial z}\right)\right]$$

$$+\hat{a}_y\left[-j\omega \bar{A}_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 \bar{A}_x}{\partial x \partial y} + \frac{\partial^2 \bar{A}_y}{\partial y^2} + \frac{\partial^2 \bar{A}_z}{\partial y \partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial \bar{F}_x}{\partial z} - \frac{\partial \bar{F}_z}{\partial x}\right)\right]$$

$$+\hat{a}_z\left[-j\omega A_z - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 \bar{A}_x}{\partial x \partial z} + \frac{\partial^2 \bar{A}_y}{\partial y \partial z} + \frac{\partial^2 \bar{A}_z}{\partial z^2}\right) - \frac{1}{\varepsilon}\left(\frac{\partial \bar{F}_y}{\partial x} - \frac{\partial \bar{F}_x}{\partial y}\right)\right] \quad (1)$$

$$\bar{E} = \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z \quad - (2)$$

Comparing (1) and (2) we get

$$E_z = \left[-j\omega A_z - \frac{j}{\omega\mu\varepsilon}\left(\cancelto{0}{\frac{\partial^2 \bar{A}_x}{\partial x \partial z}} + \cancelto{0}{\frac{\partial^2 \bar{A}_y}{\partial y \partial z}} + \frac{\partial^2 A_z}{\partial z^2}\right) - \frac{1}{\varepsilon}\left(\frac{\partial \bar{F}_y}{\partial x} - \frac{\partial \bar{F}_x}{\partial y}\right)\right] \quad (3)$$

Applying the above condition

$$E_z = \left[-j\omega A_z - \frac{j}{\omega\mu\varepsilon} \frac{\partial^2 A_z}{\partial z^2} \right]$$

$$= \frac{-j}{\omega\mu\varepsilon} \left[\frac{\partial^2 \bar{A}_z}{\partial z^2} + \frac{\omega\mu\varepsilon}{j} j\omega \bar{A}_z \right]$$

$$= \frac{-j}{\omega\mu\varepsilon} \left[\frac{\partial^2 \bar{A}_z}{\partial z^2} + \omega^2\mu\varepsilon \bar{A}_z \right]$$

$$= -\frac{j}{\omega\mu\varepsilon} \left[\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon \right] \bar{A}_z$$

This is true when

$$\bar{A}_z(x,y,z) = A_z^+(x,y)\, e^{-j\beta z} + A_z^-(x,y)\, e^{j\beta z}$$

From equation (6.43) we have.

$$H_z = \left[-j\omega F_z - \frac{j}{\omega\mu\varepsilon} \frac{\partial^2 F_z}{\partial z^2} \right]$$

$$= -\frac{j}{\omega\mu\varepsilon} \left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon \right) \bar{F}_z = 0$$

This is true when

$$\bar{F}_z(x,y,z) = \bar{F}_z^+(x,y)\, e^{-j\beta z} + \bar{F}_z^-(x,y)\, e^{j\beta z} \quad (4)$$

Now comparing eqn (1) & (2) we get

$$E_x = \left[-j\omega \cancelto{0}{A_x} - \frac{j}{\omega\mu\varepsilon} \left(\cancelto{0}{\frac{\partial^2 A_x}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 A_y}{\partial x \partial y}} + \frac{\partial^2 A_z}{\partial x \partial z} \right) - \frac{1}{\varepsilon} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \right]$$

L(4)

Applying the conditions to eqn (4)

$$E_x = \left[\frac{-j}{\omega\mu\varepsilon} \frac{\partial^2 \bar{A}_z}{\partial x \partial z} - \frac{1}{\varepsilon} \frac{\partial \bar{F}_z}{\partial y} \right]$$

$$E_x = \left[-\frac{j}{\omega\mu\varepsilon} \frac{\partial}{\partial x} \frac{\partial}{\partial z} (A_z^+ e^{-j\beta z} + A_z^- e^{j\beta z}) - \frac{1}{\varepsilon} \frac{\partial}{\partial y} (F_z^+ e^{-j\beta z} + F_z^- e^{j\beta z}) \right]$$

$$E_x = \left[\frac{-j}{\omega\mu\varepsilon} (-j\beta) \frac{\partial}{\partial x} A_z^+ e^{-j\beta z} - \frac{j}{\omega\mu\varepsilon} (j\beta) \frac{\partial}{\partial x} A_z^- e^{j\beta z} - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^+ e^{-j\beta z} - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^- e^{j\beta z} \right]$$

Separating above equation into forward & backward wave

$$E_x = \left[-\frac{\beta}{\omega\mu\varepsilon} \frac{\partial}{\partial x} A_z^+ e^{-j\beta z} - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^+ e^{-j\beta z} \right] + \left[\frac{\beta}{\omega\mu\varepsilon} \frac{\partial A_z^-}{\partial x} e^{j\beta z} - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^- e^{j\beta z} \right]$$

$$E_x = \left[-\frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu\varepsilon} \frac{\partial}{\partial x} A_z^+ e^{-j\beta z} - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^+ e^{-j\beta z} \right] + \left[\frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu\varepsilon} \frac{\partial}{\partial x} A_z^- e^{j\beta z} - \frac{1}{\varepsilon} \frac{\partial F_z^-}{\partial y} e^{j\beta z} \right]$$

$$= \overbrace{\left[-\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial}{\partial x} A_z^+ - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^+ \right] e^{-j\beta z}}^{E_x^+} + \overbrace{\left[\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial}{\partial x} A_z^- - \frac{1}{\varepsilon} \frac{\partial}{\partial y} F_z^- \right] e^{j\beta z}}^{E_x^-}$$

III^{ly}

$$E_y = \left(-\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial \bar{A}_z^+}{\partial y} + \frac{1}{\varepsilon} \frac{\partial \bar{F}_z^+}{\partial x} \right) e^{-j\beta z} + \left(\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial \bar{A}_z^-}{\partial y} + \frac{1}{\varepsilon} \frac{\partial F_z^-}{\partial x} \right) e^{j\beta z} = E_y^+ + E_y^-$$

~~Applying the conditions, we get~~

Calculating magnetic field H_x and H_y

From equation 6-43 we have

$$H_x = \left[-j\omega \cancelto{0}{\bar{F}_x} - \frac{j}{\omega\mu\varepsilon} \left(\cancelto{0}{\frac{\partial^2 \bar{F}_x}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 \bar{F}_y}{\partial x \partial y}} + \frac{\partial^2 \bar{F}_z}{\partial x \partial z} \right) + \frac{1}{\mu} \left(\frac{\partial A_z}{\partial y} - \cancelto{0}{\frac{\partial A_y}{\partial z}} \right) \right]$$

Applying the conditions given we get

$$H_x = \frac{-j}{\omega\mu\varepsilon} \frac{\partial^2 \bar{F}_z}{\partial x \partial z} + \frac{1}{\mu} \frac{\partial A_z}{\partial y}$$

Applying the condition

$$\bar{F}_z = F_z^+ e^{-j\beta z} + F_z^- e^{j\beta z}$$

$$\bar{A}_z = A_z^+ e^{-j\beta z} + A_z^- e^{j\beta z}$$

we get

$$H_x = \frac{-j}{\omega\mu\varepsilon} \left[\frac{\partial}{\partial x} \frac{\partial}{\partial z} \left(F_z^+ e^{-j\beta z} + F_z^- e^{j\beta z} \right) \right] + \frac{1}{\mu} \frac{\partial}{\partial y} \left(A_z^+ e^{-j\beta z} + A_z^- e^{j\beta z} \right)$$

$$= \frac{-j}{\omega\mu\varepsilon} \left[(-j\beta) \frac{\partial}{\partial x} F_z^+ e^{-j\beta z} \right] - \left[\frac{j(j\beta)}{\omega\mu\varepsilon} \frac{\partial}{\partial x} F_z^- e^{j\beta z} \right]$$
$$+ \frac{1}{\mu} \frac{\partial}{\partial y} A_z^+ e^{-j\beta z} + \frac{1}{\mu} \frac{\partial}{\partial y} A_z^- e^{j\beta z}$$

$$= \left[-\frac{\beta}{\omega\mu\varepsilon} \frac{\partial}{\partial x} \bar{F}_z^+ e^{-j\beta z} + \frac{1}{\mu} \frac{\partial}{\partial y} \bar{A}_z^+ e^{-j\beta z} \right]$$

$$+ \frac{\beta}{\omega\mu\varepsilon} \frac{\partial}{\partial x} \bar{F}_z^- e^{j\beta z} + \frac{1}{\mu} \frac{\partial}{\partial y} \bar{A}_z^- e^{+j\beta z}$$

$$= \left[-\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial}{\partial x} \bar{F}_z^+ + \frac{1}{\mu} \frac{\partial}{\partial y} \bar{A}_z^+ \right] e^{-j\beta z}$$

$$+ \left[\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial}{\partial x} \bar{F}_z^- + \frac{1}{\mu} \frac{\partial}{\partial y} \bar{A}_z^- \right] e^{j\beta z}$$

$$= \overbrace{-\sqrt{\frac{\varepsilon}{\mu}} \left[+ \frac{1}{\varepsilon} \frac{\partial}{\partial x} \bar{F}_z^+ - \frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial}{\partial y} \bar{A}_z^+ \right] e^{-j\beta z}}^{\bar{H}_x^+}$$

$$\underbrace{+ \sqrt{\frac{\varepsilon}{\mu}} \left[\frac{1}{\varepsilon} \frac{\partial \bar{F}_z^-}{\partial x} + \frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial \bar{A}_z^-}{\partial y} \right] e^{j\beta z}}_{\bar{H}_x^-}$$

III^{ly} $$H_y = \sqrt{\frac{\varepsilon}{\mu}} \left(-\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial \bar{A}_z^+}{\partial x} - \frac{1}{\varepsilon} \frac{\partial \bar{F}_z^+}{\partial y} \right) e^{-j\beta z}$$

$$- \sqrt{\frac{\varepsilon}{\mu}} \left(\frac{1}{\sqrt{\mu\varepsilon}} \frac{\partial \bar{A}_z^-}{\partial x} - \frac{1}{\varepsilon} \frac{\partial \bar{F}_z^-}{\partial y} \right) e^{j\beta z}$$

$$= \bar{H}_y^+ + \bar{H}_y^-$$

$$H_y^- = \sqrt{\frac{\varepsilon}{\mu}}\, E_x^+ - \sqrt{\frac{\varepsilon}{\mu}}\, E_x^-$$

$$Z_\omega^+ = \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_\omega^- = -\frac{E_x^-}{H_y^-} = \frac{E_y^-}{H_x^-} = \sqrt{\frac{\mu}{\varepsilon}}$$

|||ly you can solve the other two conditions.

$$\left.\begin{array}{ll} \text{II} & \bar{A}_z = 0 \\ \text{III} & \bar{F}_z = 0 \end{array}\right\} \quad \text{Refer T.B.}$$

Example 6-1. Using (6-41) and (6-43) derive expressions for the **E** and **H** fields, in terms of the components of the **A** and **F** potentials, that are TEM to the z direction (TEM^z).

Solution. It is apparent by examining (6-41) and (6-43) that TEM^z ($E_z = H_z = 0$) modes can be obtained by any of the following three combinations.

1. Letting

$$A_x = A_y = F_x = F_y = 0 \qquad A_z \neq 0 \qquad F_z \neq 0 \qquad \partial/\partial x \neq 0 \qquad \partial/\partial y \neq 0$$

For this combination, according to (6-41)

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon\right)A_z = 0$$

provided

$$A_z(x, y, z) = A_z^+(x, y)e^{-j\beta z} + A_z^-(x, y)e^{+j\beta z}$$

Similarly according to (6-43)

$$H_z = -j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial z^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon\right)F_z = 0$$

provided

$$F_z(x, y, z) = F_z^+(x, y)e^{-j\beta z} + F_z^-(x, y)e^{+j\beta z}$$

Also according to (6-41) and (6-43)

$$\begin{aligned}
\mathbf{E} = \hat{a}_x&\left[-j\omega A_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\,\partial y} + \frac{\partial^2 A_z}{\partial x\,\partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\right] \\
+\hat{a}_y&\left[-j\omega A_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x\,\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\,\partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\right] \\
+\hat{a}_z&\left[-j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x\,\partial z} + \frac{\partial^2 A_y}{\partial y\,\partial z} + \frac{\partial^2 A_z}{\partial z^2}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right]
\end{aligned} \tag{6-41}$$

$$\mathbf{H} = \hat{a}_x\left[-j\omega F_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x\,\partial y} + \frac{\partial^2 F_z}{\partial x\,\partial z}\right) + \frac{1}{\mu}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]$$
$$+\hat{a}_y\left[-j\omega F_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x\,\partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y\,\partial z}\right) + \frac{1}{\mu}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$
$$+\hat{a}_z\left[-j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x\,\partial z} + \frac{\partial^2 F_y}{\partial y\,\partial z} + \frac{\partial^2 F_z}{\partial z^2}\right) + \frac{1}{\mu}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \quad (6\text{-}43)$$

$$E_x = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial x} - \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y}\right)e^{-j\beta z} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial x} - \frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial y}\right)e^{+j\beta z} = E_x^+ + E_x^-$$

$$E_y = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial y} + \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial x}\right)e^{-j\beta z} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial y} + \frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial x}\right)e^{+j\beta z} = E_y^+ + E_y^-$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial y} + \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial x}\right)e^{-j\beta z}$$
$$+\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial y} + \frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial x}\right)e^{+j\beta z} = H_x^+ + H_x^-$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu}}\,(E_y^+) + \sqrt{\frac{\varepsilon}{\mu}}\,(E_y^-)$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial x} - \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y}\right)e^{-j\beta z}$$
$$-\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial x} - \frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial y}\right)e^{+j\beta z} = H_y^+ + H_y^-$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\,(E_x^+) - \sqrt{\frac{\varepsilon}{\mu}}\,(E_x^-)$$

Also

$$Z_w^+ = \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_w^- = -\frac{E_x^-}{H_y^-} = \frac{E_y^-}{H_x^-} = \sqrt{\frac{\mu}{\varepsilon}}$$

2. Letting

$$A_x = A_y = A_z = F_x = F_y = 0 \qquad F_z \neq 0 \qquad \partial/\partial x \neq 0 \qquad \partial/\partial y \neq 0$$

For this combination, according to (6-41) and (6-43)

$$E_z = 0$$

$$H_z = -j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial z^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon\right)F_z = 0$$

provided

$$F_z(x, y, z) = F_z^+(x, y)e^{-j\beta z} + F_z^-(x, y)e^{+j\beta z}$$

Also according to (6-41) and (6-43)

$$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y}e^{-j\beta z} - \frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial y}e^{+j\beta z} = E_x^+ + E_x^-$$

$$E_y = +\frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial x}e^{-j\beta z} + \frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial x}e^{+j\beta z} = E_y^+ + E_y^-$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial x}\right)e^{-j\beta z} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial x}\right)e^{+j\beta z} = H_x^+ + H_x^-$$

$$= -\sqrt{\frac{\varepsilon}{\mu}}(E_y^+) + \sqrt{\frac{\varepsilon}{\mu}}(E_y^-)$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y}\right)e^{-j\beta z} - \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_z^-}{\partial y}\right)e^{+j\beta z} = H_y^+ + H_y^-$$

$$= \sqrt{\frac{\varepsilon}{\mu}}(E_x^+) - \sqrt{\frac{\varepsilon}{\mu}}(E_x^-)$$

Also

$$Z_w^+ = \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_w^- = -\frac{E_x^-}{H_y^-} = \frac{E_y^-}{H_x^-} = \sqrt{\frac{\mu}{\varepsilon}}$$

3. Letting

$$A_x = A_y = F_x = F_y = F_z = 0 \qquad A_z \neq 0 \qquad \partial/\partial x \neq 0 \qquad \partial/\partial y \neq 0$$

For this combination, according to (6-41) and (6-43)

$$H_z = 0$$

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon\right)A_z = 0$$

provided

$$A_z(x, y, z) = A_z^+(x, y)e^{-j\beta z} + A_z^-(x, y)e^{+j\beta z}$$

Also according to (6-41) and (6-43)

$$E_x = -\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial x}e^{-j\beta z} + \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial x}e^{+j\beta z} = E_x^+ + E_x^-$$

$$E_y = -\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial y}e^{-j\beta z} + \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial y}e^{+j\beta z} = E_y^+ + E_y^-$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial y}\right)e^{-j\beta z} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial y}\right)e^{+j\beta z} = H_x^+ + H_x^-$$

$$= -\sqrt{\frac{\varepsilon}{\mu}}(E_y^+) + \sqrt{\frac{\varepsilon}{\mu}}(E_y^-)$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial y}\right)e^{-j\beta z} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial y}\right)e^{+j\beta z} = H_x^+ + H_x^-$$

$$= -\sqrt{\frac{\varepsilon}{\mu}}(E_y^+) + \sqrt{\frac{\varepsilon}{\mu}}(E_y^-)$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial x}\right)e^{-j\beta z} - \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^-}{\partial x}\right)e^{+j\beta z} = H_y^+ + H_y^-$$

$$= \sqrt{\frac{\varepsilon}{\mu}}(E_x^+) - \sqrt{\frac{\varepsilon}{\mu}}(E_x^-)$$

Also

$$Z_w^+ = \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_w^- = -\frac{E_x^-}{H_y^-} = \frac{E_y^-}{H_x^-} = \sqrt{\frac{\mu}{\varepsilon}}$$

SUMMARY

From the results of Example 6-1, it is evident that TEM^z modes can be obtained by any of the following three combinations:

TEM^z

$$A_x = A_y = F_x = F_y = 0 \qquad \partial/\partial x \neq 0 \qquad \partial/\partial y \neq 0 \tag{6-44}$$

$$A_z = A_z^+(x, y)e^{-j\beta z} + A_z^-(x, y)e^{+j\beta z} \tag{6-44a}$$

$$F_z = F_z^+(x, y)e^{-j\beta z} + F_z^-(x, y)e^{+j\beta z} \tag{6-44b}$$

$$A_x = A_y = A_z = F_x = F_y = 0 \qquad \partial/\partial x \neq 0 \qquad \partial/\partial y \neq 0 \tag{6-45}$$

$$F_z = F_z^+(x, y)e^{-j\beta z} + F_z^-(x, y)e^{+j\beta z} \tag{6-45a}$$

$$A_x = A_y = F_x = F_y = F_z = 0 \qquad \partial/\partial x \neq 0 \qquad \partial/\partial y \neq 0 \tag{6-46}$$

$$A_z = A_z^+(x, y)e^{-j\beta z} + A_z^-(x, y)e^{+j\beta z} \tag{6-46a}$$

A similar procedure can be used to derive TEM modes in other directions such as TEM^x and TEM^y.

B. CYLINDRICAL COORDINATE SYSTEM

To derive expressions for TEM modes in a cylindrical coordinate system, a procedure similar to that in the rectangular coordinate system can be used. When (6-34)

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\mathbf{A}) - \frac{1}{\varepsilon}\nabla\times\mathbf{F} \tag{6-47}$$

and (6-35)

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla\times\mathbf{A} - j\omega\mathbf{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\mathbf{F}) \tag{6-48}$$

are expanded using

$$\mathbf{A}(\rho,\phi,z) = \hat{a}_\rho A_\rho(\rho,\phi,z) + \hat{a}_\phi A_\phi(\rho,\phi,z) + \hat{a}_z A_z(\rho,\phi,z) \tag{6-49a}$$

$$\mathbf{F}(\rho,\phi,z) = \hat{a}_\rho F_\rho(\rho,\phi,z) + \hat{a}_\phi F_\phi(\rho,\phi,z) + \hat{a}_z F_z(\rho,\phi,z) \tag{6-49b}$$

as solutions, they can be written as

$$\begin{aligned}
\mathbf{E} = {} & \hat{a}_\rho\left\{-j\omega A_\rho - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\right] - \frac{1}{\varepsilon}\left(\frac{1}{\rho}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right)\right\} \\
& + \hat{a}_\phi\left\{-j\omega A_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\right]\right. \\
& \qquad \left. - \frac{1}{\varepsilon}\left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\right\} \\
& + \hat{a}_z\left\{-j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\right]\right. \\
& \qquad \left. - \frac{1}{\varepsilon}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\phi) - \frac{\partial F_\rho}{\partial\phi}\right]\right\}
\end{aligned} \tag{6-50}$$

$$\mathbf{H} = \hat{a}_\rho\left\{-j\omega F_\rho - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}\right] + \frac{1}{\mu}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\right\}$$
$$+\hat{a}_\phi\left\{-j\omega F_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}\right] + \frac{1}{\mu}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\right\}$$
$$+\hat{a}_z\left\{-j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}\right] + \frac{1}{\mu}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\right\} \tag{6-51}$$

Example 6-2. Using (6-50) and (6-51) derive expressions for the **E** and **H** fields, in terms of the components of the **A** and **F** potentials, that are TEM to the ρ direction (TEM$^\rho$).

Solution. It is apparent by examining (6-50) and (6-51) that TEM$^\rho$ ($E_\rho = H_\rho = 0$) modes can be obtained by any of the following three combinations:

1. Letting

$$A_\phi = A_z = F_\phi = F_z = 0 \qquad A_\rho \neq 0 \qquad F_\rho \neq 0 \qquad \partial/\partial\phi \neq 0 \qquad \partial/\partial z \neq 0$$

For this combination, according to (6-50) and (6-51)

$$E_\rho = -j\omega A_\rho - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] = -j\frac{1}{\omega\mu\varepsilon}\left\{\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] + \omega^2\mu\varepsilon A_\rho\right\}$$
$$= -j\frac{1}{\omega\mu\varepsilon}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial A_\rho}{\partial\rho} + \frac{A_\rho}{\rho}\right) + \omega^2\mu\varepsilon A_\rho\right]$$
$$= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial\rho} - \frac{A_\rho}{\rho^2} + \omega^2\mu\varepsilon A_\rho\right)$$
$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \beta^2\right)A_\rho = 0$$

provided

$$A_\rho(\rho, \phi, z) = A_\rho^+(\phi, z)H_1^{(2)}(\beta\rho) + A_\rho^-(\phi, z)H_1^{(1)}(\beta\rho)$$

Also

$$\begin{aligned}
H_\rho &= -j\omega F_\rho - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)\right] \\
&= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2} + \beta^2\right)F_\rho = 0
\end{aligned}$$

provided

$$F_\rho(\rho, \phi, z) = F_\rho^+(\phi, z)H_1^{(2)}(\beta\rho) + F_\rho^-(\phi, z)H_1^{(1)}(\beta\rho)$$

In addition

$$\begin{aligned}
E_\phi &= -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] - \frac{1}{\varepsilon}\frac{\partial F_\rho}{\partial z} \\
E_z &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] - \frac{1}{\varepsilon}\left(-\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi}\right) \\
H_\phi &= -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)\right] + \frac{1}{\mu}\frac{\partial A_\rho}{\partial z} \\
H_z &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)\right] + \frac{1}{\mu}\left(-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)
\end{aligned}$$

2. Letting

$$A_\rho = A_\phi = A_z = F_\phi = F_z = 0 \qquad F_\rho \neq 0 \qquad \partial/\partial\phi \neq 0 \qquad \partial/\partial z \neq 0$$

For this combination, according to (6-50) and (6-51)

$$E_\rho = 0$$

$$H_\rho = -j\omega F_\rho - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)\right]$$

$$= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \beta^2\right)F_\rho = 0$$

provided

$$F_\rho(\rho, \phi, z) = F_\rho^+(\phi, z)H_1^{(2)}(\beta\rho) + F_\rho^-(\phi, z)H_1^{(1)}(\beta\rho)$$

In addition

$$E_\phi = -\frac{1}{\varepsilon}\frac{\partial F_\rho}{\partial z}$$

$$E_z = -\frac{1}{\varepsilon}\left(-\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi}\right)$$

$$H_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)\right]$$

$$H_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)\right]$$

3. Letting

$$A_\phi = A_z = F_\rho = F_\phi = F_z = 0 \qquad A_\rho \neq 0 \qquad \partial/\partial\phi \neq 0 \qquad \partial/\partial z \neq 0$$

For this combination, according to (6-50) and (6-51)

$$H_\rho = 0$$

$$\begin{aligned} E_\rho &= -j\omega A_\rho - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] \\ &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \beta^2\right)A_\rho = 0 \end{aligned}$$

provided

$$A_\rho(\rho, \phi, z) = A_\rho^+(\phi, z)H_1^{(2)}(\beta\rho) + A_\rho^-(\phi, z)H_1^{(1)}(\beta\rho)$$

In addition

$$\begin{aligned} E_\phi &= -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] \\ E_z &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)\right] \\ H_\phi &= \frac{1}{\mu}\left(\frac{\partial A_\rho}{\partial z}\right) \\ H_z &= \frac{1}{\mu}\left(-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right) \end{aligned}$$

SUMMARY

From the results of Example 6-2, it is evident that TEM^{ρ} modes can be obtained by any of the following three combinations:

$$\boxed{\begin{aligned} &A_\phi = A_z = F_\phi = F_z = 0 \qquad \partial/\partial\phi \neq 0 \qquad \partial/\partial z \neq 0 \\ &A_\rho(\rho,\phi,z) = A_\rho^{+}(\phi,z)H_1^{(2)}(\beta\rho) + A_\rho^{-}(\phi,z)H_1^{(1)}(\beta\rho) \\ &F_\rho(\rho,\phi,z) = F_\rho^{+}(\phi,z)H_1^{(2)}(\beta\rho) + F_\rho^{-}(\phi,z)H_1^{(1)}(\beta\rho) \end{aligned}} \qquad \begin{aligned} &(6\text{-}52) \\ &(6\text{-}52\text{a}) \\ &(6\text{-}52\text{b}) \end{aligned}$$

$$\boxed{\begin{aligned} &A_\rho = A_\phi = A_z = F_\phi = F_z = 0 \qquad \partial/\partial\phi \neq 0 \qquad \partial/\partial z \neq 0 \\ &F_\rho(\rho,\phi,z) = F_\rho^{+}(\phi,z)H_1^{(2)}(\beta\rho) + F_\rho^{-}(\phi,z)H_1^{(1)}(\beta\rho) \end{aligned}} \qquad \begin{aligned} &(6\text{-}53) \\ &(6\text{-}53\text{a}) \end{aligned}$$

$$\boxed{\begin{aligned} &A_\phi = A_z = F_\rho = F_\phi = F_z = 0 \qquad \partial/\partial\phi \neq 0 \qquad \partial/\partial z \neq 0 \\ &A_\rho(\rho,\phi,z) = A_\rho^{+}(\phi,z)H_1^{(2)}(\beta\rho) + A_\rho^{-}(\phi,z)H_1^{(1)}(\beta\rho) \end{aligned}} \qquad \begin{aligned} &(6\text{-}54) \\ &(6\text{-}54\text{a}) \end{aligned}$$

A similar procedure can be used to derive TEM modes in other directions such as TEM^{ϕ} and TEM^{z}.

6.5.2 Transverse Magnetic Modes: Source-Free Region

Often we seek solutions of higher-order modes, other than transverse electromagnetic (TEM). Some of the higher-order modes, often required to satisfy boundary conditions, are designated as transverse magnetic (TM) and transverse electric (TE). Classical examples of the need for TM and TE modes are modes of propagation in waveguides [2].

Transverse magnetic modes (often also known as transverse magnetic fields) are field configurations whose magnetic field components lie in a plane that is transverse to a given direction. That direction is often chosen to be the path of wave propagation. For example, if the desired fields are TM to z (TM^z), this implies that $H_z = 0$. The other two magnetic field components (H_x and H_y) and three electric field components (E_x, E_y, and E_z) may or may not all exist.

By examining (6-43) and (6-51) it is evident that *to derive the field expressions that are TM to a given direction, independent of the coordinate system, it is sufficient to let the vector potential* **A** *have only a component in the direction in which the fields are desired to be TM. The remaining components of* **A** *as well as all of* **F** *are set equal to zero.*

A. RECTANGULAR COORDINATE SYSTEM

$\underline{\mathrm{TM}^z}$

To demonstrate the aforementioned procedure, let us assume that we wish to derive field expressions that are TM to z (TM^z). To accomplish this, we let

$$\boxed{\begin{aligned} \mathbf{A} &= \hat{a}_z A_z(x, y, z) \qquad (6\text{-}55a) \\ \mathbf{F} &= 0 \qquad (6\text{-}55b) \end{aligned}}$$

The vector potential **A** must satisfy (6-30) which reduces from a vector wave equation to a scalar wave equation

$$\nabla^2 A_z(x, y, z) + \beta^2 A_z(x, y, z) = 0 \tag{6-56}$$

Since (6-56) is of the same form as (3-20a), its solution using the *separation of variables method* can be written according to (3-23) as

$$A_z(x, y, z) = f(x)g(y)h(z) \tag{6-57}$$

The solutions of $f(x)$, $g(y)$, and $h(z)$ take the forms given by (3-28a) through (3-30b). The most appropriate forms for $f(x)$, $g(y)$, and $h(z)$ much be chosen judiciously to reduce the complexity of the problem, and they will depend on the configuration of the problem. For the rectangular waveguide of Figure 3-2, for example, the most appropriate forms for $f(x)$, $g(y)$, and $h(z)$ are those given, respectively, by (3-28b), (3-29b), and (3-30a). Thus for the rectangular waveguide, (6-57) can be written as

$$\begin{aligned} A_z(x, y, z) = {} & [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)][C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \\ & \times \left(A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right) \end{aligned} \tag{6-58}$$

where

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \varepsilon \tag{6-58a}$$

Once A_z is found, the next step is to use (6-41) and (6-43) to find the **E** and **H** field components. Doing this, it can be shown that by using (6-55a) and (6-55b) we can reduce (6-41) and (6-43) to

TM^z Rectangular Coordinate System

$$\begin{aligned}
E_x &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x\,\partial z} & H_x &= \frac{1}{\mu}\frac{\partial A_z}{\partial y} \\
E_y &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y\,\partial z} & H_y &= -\frac{1}{\mu}\frac{\partial A_z}{\partial x} \\
E_z &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z & H_z &= 0
\end{aligned} \tag{6-59}$$

which satisfy the definition of TM^z (i.e., $H_z = 0$).

For the specific example for which the solution of A_z as given by (6-58) is applicable, the unknown constants C_1, D_1, C_2, D_2, A_3, B_3, β_x, β_y, and β_z can be evaluated by substituting A_z of (6-58) into the expressions for **E** and **H** in (6-59) and enforcing the appropriate boundary conditions on the **E** and **H** field components. This will be demonstrated in Chapter 8, and elsewhere, where specific problem configurations are attempted. Following these or similar procedures should lead to the solution of the problem in question.

Expressions for the **E** and **H** field components that are TM^x and TM^y are given, respectively, by

TM^x Rectangular Coordinate System

Let

$$\boxed{\begin{aligned} \mathbf{A} &= \hat{a}_x A_x(x, y, z) \qquad (6\text{-}60a)\\ \mathbf{F} &= 0 \qquad (6\text{-}60b)\end{aligned}}$$

Then

$$\boxed{\begin{aligned} E_x &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + \beta^2\right)A_x & H_x &= 0\\ E_y &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_x}{\partial x\,\partial y} & H_y &= \frac{1}{\mu}\frac{\partial A_x}{\partial z}\\ E_z &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_x}{\partial x\,\partial z} & H_z &= -\frac{1}{\mu}\frac{\partial A_x}{\partial y}\end{aligned}} \qquad (6\text{-}61)$$

where A_x must satisfy the scalar wave equation

$$\nabla^2 A_x(x, y, z) + \beta^2 A_x(x, y, z) = 0 \qquad (6\text{-}62)$$

TM^y Rectangular Coordinate System

Let

$$\boxed{\begin{aligned} &\mathbf{A} = \hat{a}_y A_y(x, y, z) && (6\text{-}63a) \\ &\mathbf{F} = 0 && (6\text{-}63b) \end{aligned}}$$

Then

$$\boxed{\begin{aligned} E_x &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_y}{\partial x\,\partial y} & H_x &= -\frac{1}{\mu}\frac{\partial A_y}{\partial z} \\ E_y &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2} + \beta^2\right)A_y & H_y &= 0 \\ E_z &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_y}{\partial y\,\partial z} & H_z &= \frac{1}{\mu}\frac{\partial A_y}{\partial x} \end{aligned}} \qquad (6\text{-}64)$$

where A_y must satisfy the scalar wave equation of

$$\nabla^2 A_y(x, y, z) + \beta^2 A_y(x, y, z) = 0 \qquad (6\text{-}65)$$

The derivations of (6-61) and (6-64) are left to the reader as end of chapter assignments.

B. CYLINDRICAL COORDINATE SYSTEM

In terms of complexity, the next higher-order coordinate system is that of the cylindrical coordinate system. We will derive expressions that will be valid for TM^z. TM^ρ and TM^ϕ are more difficult and are not usually utilized. Therefore they will not be attempted here. The procedure for TM^z in a cylindrical coordinate system is the same as that used for the rectangular coordinate system, as outlined previously in this section.

$\underline{TM^z}$

To accomplish this, let

$$\boxed{\begin{aligned} \mathbf{A} &= \hat{a}_z A_z(\rho, \phi, z) \\ \mathbf{F} &= 0 \end{aligned}} \qquad \begin{aligned} &(6\text{-}66a) \\ &(6\text{-}66b) \end{aligned}$$

The vector potential **A** must satisfy (6-30) with $\mathbf{J} = 0$ which reduces from its vector form to the scalar wave equation

$$\nabla^2 A_z(\rho, \phi, z) + \beta^2 A_z(\rho, \phi, z) = 0 \tag{6-67}$$

Since (6-67) is of the same form as (3-54c), its solution using the *separation of variables method* can be written according to (3-57) as

$$A_z(\rho, \phi, z) = f(\rho) g(\phi) h(z) \tag{6-68}$$

The solutions of $f(\rho)$, $g(\phi)$, and $h(z)$ take the forms given by (3-67a) through (3-69b). The most appropriate forms for $f(\rho)$, $g(\phi)$, and $h(z)$ must be chosen judiciously to reduce the complexity of the problem, and they will depend upon the configuration of the problem. For the cylindrical waveguide of Figure 3-5, for example, the most appropriate forms for $f(\rho)$, $g(\phi)$, and $h(z)$ are those given, respectively, by (3-67a), (3-68b), and (3-69a). Thus for the cylindrical waveguide, (6-68) can be written as

$$\begin{aligned} A_z(\rho,\phi,z) &= \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \\ &\quad \times \left(A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right) \end{aligned} \tag{6-69}$$

where

$$\beta_\rho^2 + \beta_z^2 = \beta^2 \tag{6-69a}$$

Once A_z is found, the next step is to use (6-50) and (6-51) to find the **E** and **H** field components. Then we can show that by using (6-66a) and (6-66b), (6-50) and (6-51) can be reduced to

TM^z Cylindrical Coordinate System

$$\begin{aligned} E_\rho &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\,\partial z} & H_\rho &= \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} \\ E_\phi &= -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z} & H_\phi &= -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} \\ E_z &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z & H_z &= 0 \end{aligned} \tag{6-70}$$

which also satisfies the TM^z definition (i.e., $H_z = 0$).

Solution of the inhomogeneous vector Potential wave equation

The inhomogeneous vector wave equation can be written as:-

$$\nabla^2 \bar{A} + \beta^2 \bar{A} = -\mu \bar{J} \qquad -(1)$$

&

$$\nabla^2 \bar{F} + \beta^2 \bar{F} = -\varepsilon \bar{M} \qquad -(2)$$

If we assume the current density to vary only along the z direction. The only component that exists is the $\bar{A}_z$ component. Assuming this we can write (2) as:

$$\nabla^2 \bar{A}_z + \beta^2 \bar{A}_z = -\mu \bar{J}_z \qquad -(3)$$

For a source free region (3) becomes

$$\nabla^2 \bar{A}_z + \beta^2 \bar{A}_z = 0 \qquad -(4)$$

In the limit the source is a point, it requires $\bar{A}_z$ to be a function of r the radial distance in spherical coordinate system.

Thus equation (4) can be written as.

$$\frac{d^2 A_z(r)}{\partial} \quad \nabla^2 A_z(r) + k^2 A_z(r) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \bar{A}_z(r)}{\partial r} \right) + k^2 \bar{A}_z(r) = 0 \qquad -(5)$$

This can be solved as:

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z(r)}{\partial r} \right) \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z(r)}{\partial r} \right)$$

$$= \frac{1}{r^2}\left[r^2 \frac{\partial^2 A_z(r)}{\partial r^2} + 2r \frac{\partial A_z(r)}{\partial r} \right]$$

$$= \frac{\partial^2 A_z(r)}{\partial r^2} + \frac{2}{r} \frac{\partial A_z(r)}{\partial r}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_z(r)}{\partial r}\right) \Rightarrow \frac{\partial^2 A_z(r)}{\partial r^2} + \frac{2}{r}\frac{\partial A_z(r)}{\partial r} \quad -(6)$$

Substituting (6) into (5) we get

$$\nabla^2 A_z(r) + \beta^2 A_z(r) = \frac{\partial^2 A_z(r)}{\partial r^2} + \frac{2}{r}\frac{\partial A_z(r)}{\partial r} + k^2 A_z(r) = 0$$

This differential equation has two independent solutions

$$\bar{A}_{z_1} = C_1\frac{e^{-j\beta r}}{r} \qquad \bar{A}_{z_2} = C_2\frac{e^{j\beta r}}{r}$$

These represents the traveling wave in the outward & inward direction.

If we assume that the source is placed at the origin with the radiated fields traveling in the outward radial direction. Therefore we choose the solution.

$$\bar{A}_z = \bar{A}_{z_1} = C_1\frac{e^{-j\beta r}}{r} e^{j\omega t} \quad -(7)$$

For the static case ($\omega = 0$, $\beta = 0$) & the above equation reduces to

$$\bar{A}_z = \frac{C_1}{r} \quad -(8)$$

which is solution of wave equation when $\beta = 0$.
At points away from the source, the time varying and static solutions of (7) and (8) differ only by the $e^{-j\beta r}$ factor, or time varying solution of (7) can be obtained by multiplying the static solution of (6) by $e^{-j\beta r}$

In the presence of source and with $\beta = 0$ the wave equation reduces to

$$\nabla^2 \bar{A}_z = -\mu \bar{J}_z \qquad -(9)$$

This equation is a poisson equation ~~ge~~ which is generally given as:

$$\nabla^2 \phi = -\frac{\rho_v}{\varepsilon} \qquad -(10)$$

& the solution of this equation is given by

$$\phi = \frac{1}{4\pi\varepsilon} \iiint_v \frac{\rho_v}{r} \, dv' \qquad -(11)$$

$r \rightarrow$ distance from any point on charge density to the observation point. Writing solution for (9) in the form of (11) we get

$$\bar{A}_z = \frac{\mu}{4\pi} \iiint_v \frac{\bar{J}_z}{r} \, dv' \qquad -(12)$$

This is the solution in the static case. The time varying solution can be obtained by multiplying the static solution by $e^{-j\beta r}$. Thus

$$\boxed{\bar{A}_z = \frac{\mu}{4\pi} \iiint_v \bar{J}_z \frac{e^{-j\beta r}}{r} \, dv'}$$

If the current densities were in the x and y directions the wave equation for each would reduce to

$$\nabla^2 \bar{A}_x + \beta^2 \bar{A}_x = -\mu \bar{J}_x \qquad -(13)$$

$$\nabla^2 \bar{A}_y + \beta^2 \bar{A}_y = -\mu \bar{J}_y \qquad -(14)$$

with solution

$$\bar{A}_x = \frac{\mu}{4\pi} \iiint_v \bar{J}_x \frac{e^{-j\beta r}}{r} \, dv'$$

$$\bar{A}_y = \frac{\mu}{4\pi} \iiint_v \bar{J}_y \frac{e^{-j\beta r}}{r} \, dv'$$

The general solution of the vector wave equation can be written as.

$$\bar{A} = \frac{\mu}{4\pi} \iiint_v \bar{J} \frac{e^{-j\beta r}}{r} \, dv'$$

If the source is removed from the origin and placed at a position represented by (x', y', z') the solution can be written as

$$\bar{A}(x,y,z) = \frac{\mu}{4\pi} \iiint_v \bar{J}(x',y',z') \frac{e^{-j\beta R}}{\bar{R}} \, dv'$$

IIIly $\bar{F}(x,y,z) = \frac{\varepsilon}{4\pi} \iiint_v M(x',y',z') \frac{e^{-j\beta R}}{R} \, dv'$

If $\bar{J}$ and $\bar{M}$ represent linear densities. the equation reduces to surface integrals.

$$\bar{A} = \frac{\mu}{4\pi} \iint_S \bar{J}_s(x', y', z') \frac{e^{-j\beta R}}{R} ds'$$

$$F = \frac{\varepsilon}{4\pi} \iint_S \bar{M}_s(x', y', z') \frac{e^{-j\beta R}}{R} ds'$$

For electric and magnetic currents $\bar{I}_e$ and $\bar{I}_m$ they reduce to.

$$\bar{A} = \frac{\mu}{4\pi} \int_c \bar{I}_e(x', y', z') \frac{e^{-j\beta R}}{R} dl'$$

$$\bar{F} = \frac{\varepsilon}{4\pi} \int_c \bar{I}_m(x', y', z') \frac{e^{-j\beta R}}{R} dl'$$