$\boxed{14.3}$

$$\frac{d^2V}{dx^2} = f(x) = -\frac{1}{\varepsilon_0}q(x),\ V(x=0) = V(x=w) = 0$$

$$\frac{d^2V}{dx^2} = 0 \Rightarrow \begin{array}{l} V_1(x) = A_1x + B_1 \\ V_2(x) = A_2x + B_2 \end{array}$$

Since $V_1(x=0) = 0 \Rightarrow V_1(x) = A_1x \quad , V_1'(x) = A_1$

$V_2(x=w) = 0 \Rightarrow V_2(x) = A_2(x-w) \ , V_2'(x) = A_2$

$$W(x') = V_1(x')V_2'(x') - V_2(x')V_1'(x') = A_1A_2x' - A_2A_1(x'-w) = A_1A_2w$$

$$G(x,x') = \begin{cases} \dfrac{A_2(x'-w)}{A_1A_2w}A_1x = \left(\dfrac{x'-w}{w}\right)x & , 0 \le x \le x' \\[2ex] \dfrac{A_1x'}{A_1A_2w}A_2(x-w) = \dfrac{x'}{w}(x-w) & , x' \le x \le w \end{cases}$$

$\boxed{14.8}$ The Green's function will be chosen to satisfy the partial differential equation of

$$\frac{\partial^2G}{\partial x^2} + \frac{\partial^2G}{\partial y^2} = \delta(x-x')(y-y') \qquad (1)$$

subject to the boundary conditions of

$$G(x=0, 0\le y\le\infty) = G(x=a, 0\le y\le\infty) = 0 \qquad (2a)$$

$$G(0\le x\le a, y=0) = 0\ ;\ G(0\le x\le a, y=\infty) = \text{Finite} \qquad (2b)$$

Choose the Green's function to satisfy the boundary conditions of (2a) at $x=0$ and $x=a$. Thus

$$G(x,y;x',y') = \sum_{m=1,2,..}^{\infty} g_m(y;x',y')\sin\left(\frac{m\pi}{a}x\right) \qquad (3)$$

Substituting (3) into (1) leads to

$$\sum_{m=1,2,..}^{\infty}\left[-\left(\frac{m\pi}{a}\right)^2 g_m(y;x',y') + \frac{d^2g_m(y;x',y')}{dy^2}\right]\sin\left(\frac{m\pi}{a}x\right) = \delta(x-x')\delta(y-y') \qquad (4)$$

Multiplying both sides of (4) by $\sin(m\pi x/a)$, integrating with respect to x from 0 to a, and using (14-48a) and (14-48b), we can write that

$$\frac{d^2g_m}{dy^2} - \left(\frac{m\pi}{a}\right)^2 g_m = \frac{2}{a}\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y') \qquad (5)$$

For the homogeneous form of (5), the two solutions can be written as

$$g_m^{(1)}(y;x',y') = A_m(x',y')e^{-(m\pi/a)y} + B_m(x',y')e^{+(m\pi/a)y} \quad , y<y' \quad (6)$$

$$g_m^{(2)}(y;x',y') = C_m(x',y')e^{-(m\pi/a)y} + D_m(x',y')e^{+(m\pi/a)y} \quad , y>y' \quad (7)$$

Now apply on (6) and (7) the boundary conditions of (2b).

$$g_m^{(1)}(y=0) = A_m + B_m = 0 \Rightarrow B_m = -A_m \tag{8}$$

$$g_m^{(2)}(y=\infty) = \text{finite} \Rightarrow D_m = 0 \tag{9}$$

Thus

$$g_m^{(1)}(y; x', y') = 2A_m(x', y')\left[\frac{e^{-(m\pi/a)y} - e^{+(m\pi/a)y}}{2}\right] = 2A_m(x', y')\sinh\left(\frac{m\pi}{a}y\right) \quad y < y' \tag{10}$$

$$g_m^{(2)}(y; x', y') = C_m(x', y')\, e^{-(m\pi/a)y} \quad y > y' \tag{11}$$

<u>Continuity of $g_m^{(1)}$ and $g_m^{(2)}$ at $y=y'$:</u>

$$2A_m(x', y')\sinh\left(\frac{m\pi}{a}y'\right) = C_m(x', y')\, e^{-\left(\frac{m\pi}{a}y'\right)} \tag{12}$$

<u>Discontinuity of $dg_m^{(1)}/dy|_{y=y'}$ and $dg_m^{(2)}/dy|_{y=y'}$:</u>

$$\left(\frac{m\pi}{a}\right)\left[-C_m e^{-(m\pi/a)y'} - 2A_m \cosh\left(\frac{m\pi}{a}y'\right)\right] = 1 \tag{13}$$

cont'd.

$\boxed{\text{14.8 cont'd.}}$ Equations (12) and (13) can be written as

$$C_m = 2A_m \sinh\left(\frac{m\pi}{a}y'\right) e^{+(m\pi/a)y'} \tag{14}$$

$$C_m e^{-(m\pi/a)y'} + 2A_m \cosh\left(\frac{m\pi}{a}y'\right) = -\frac{a}{m\pi} \tag{15}$$

Solving these two equations for A_m and C_m leads to

$$A_m = -\frac{2}{m\pi}\,\frac{1}{2\left[\cosh\left(\frac{m\pi}{a}y'\right) + \sinh\left(\frac{m\pi}{a}y'\right)\right]} \tag{16}$$

$$C_m = -\frac{2}{m\pi}\,\frac{\sinh\left(\frac{m\pi}{a}y'\right) e^{(m\pi/a)y'}}{\cosh\left(\frac{m\pi}{a}y'\right) + \sinh\left(\frac{m\pi}{a}y'\right)} \tag{17}$$

Thus (10) and (11) can be written as

$$g_m^{(1)}(y;x',y') = -\frac{2}{m\pi}\frac{\sinh\left(\frac{m\pi}{a}y\right)}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)} \quad , \; y<y' \qquad (18)$$

$$g_m^{(2)}(y;x',y') = -\frac{2}{m\pi}\frac{\sinh\left(\frac{m\pi}{a}y'\right)e^{(m\pi/a)y'}}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}e^{-(m\pi/a)y} \quad , \; y>y' \qquad (19)$$

Thus the Green's function of (3) can be expressed as

$$G(x,y;x',y') = \begin{cases} \displaystyle\sum_{m=1,2,\ldots}^{\infty} -\left(\frac{2}{m\pi}\right)\frac{1}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}\sinh\left(\frac{m\pi}{a}y\right)\sin\left(\frac{m\pi}{a}x\right) \; , & y<y' \\ \displaystyle\sum_{m=1,2,\ldots}^{\infty} -\left(\frac{2}{m\pi}\right)\frac{\sinh\left(\frac{m\pi}{a}y'\right)e^{(m\pi/a)y'}}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}e^{-\left(\frac{m\pi}{a}\right)y}\sin\left(\frac{m\pi}{a}x\right) \; , & y>y' \end{cases}$$

14.11 $\quad \dfrac{\partial^2 G}{\partial x^2}+\dfrac{\partial^2 G}{\partial y^2}+\beta^2 G = \delta(x-x')\delta(y-y') \qquad (1)$

$$G(x=0,\, 0\le y\le b) = G(x=a,\, 0\le y\le b) = 0 \qquad (1a)$$
$$G(0\le x\le a,\, y=0) = G(0\le x\le a,\, y=b) = 0 \qquad (1b)$$

$$G(x,y;x',y') = \sum_m g_m(y;x',y')\left[A_m\sin(\beta_x x)+B_m\cos(\beta_x x)\right] \qquad (2)$$

Applying (1a) on (2) leads to

$$G(x=0) = \sum_m g_m(y;x',y')\left[A_m(0)+B_m(1)\right]=0 \;\Rightarrow B_m=0$$

$$G(x=a) = \sum_m g_m(y;x',y')\left[A_m\sin(\beta_x a)\right]=0 \Rightarrow \beta_x a = \sin^{-1}(0)=m\pi \Rightarrow \beta_x=\left(\frac{m\pi}{a}\right), m=1,2,\ldots$$

Thus

$$G(x,y;x',y') = \sum_{m=1,2,\ldots}^{\infty} A_m g_m(y;x',y')\sin\left(\frac{m\pi}{a}x\right) \qquad (3)$$

Substituting (3) into (1), we can write ~~that~~

$$\sum_{m=1,2,\ldots}^{\infty} A_m\left\{-\left(\frac{m\pi}{a}\right)^2 g_m + \frac{d^2 g_m}{dy^2}+\beta^2 g_m\right\}\sin\left(\frac{m\pi}{a}x\right) = \delta(x-x')\delta(y-y')$$

$$\sum_{m=1,2,\ldots}^{\infty} A_m\left\{\frac{d^2 g_m}{dy^2}+\left[\beta^2-\left(\frac{m\pi}{a}\right)^2\right]g_m\right\}\sin\left(\frac{m\pi}{a}x\right) = \delta(x-x')\delta(y-y') \qquad (4)$$

Multiplying both sides by $\sin(n\pi x/a)$, integrating between 0 to a in x, and using (14-48a) and (14-48b), leads to

$\boxed{\text{14.11 cont'd.}}$ $\frac{a}{2}\left\{\frac{d^2 g_m}{dy^2}+\left[\beta^2-\left(\frac{m\pi}{a}\right)^2\right]g_m\right\}=\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y')$

or $\frac{d^2 g_m}{dy^2}+\left[\beta^2-\left(\frac{m\pi}{a}\right)^2\right]g_m=\frac{d^2 g_m}{dy^2}+\beta_y^2 g_m=\frac{2}{a}\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y')$, where $\beta_y^2=\beta^2-\left(\frac{m\pi}{a}\right)^2$ (5)

The two solutions for the homogeneous form of (5) are

$$g_m^{(1)}=A_1\cos(\beta_y y)+B_1\sin(\beta_y y) \qquad y<y' \qquad (6a)$$

$$g_m^{(2)}=C_1\cos[\beta_y(y-b)]+D_1\sin[\beta_y(y-b)] \qquad y>y' \qquad (6b)$$

Applying the boundary conditions of (1b) on (6a) and (6b) leads to

$$g_m^{(1)}(y=0)=A_1(1)+B_1(0)=0 \Rightarrow A_1=0$$

$$g_m^{(2)}(y=b)=C_1(1)+D_1(0)=0 \Rightarrow C_1=0$$

Thus (6a) and (6b) reduce to

$$g_m^{(1)}=B_1\sin(\beta_y y), \quad g_m^{(1)\prime}=\beta_y B_1\cos(\beta_y y), \quad y<y' \qquad (7a)$$

$$g_m^{(2)}=D_1\sin[\beta_y(y-b)], \quad g_m^{(2)\prime}=\beta_y D_1\cos[\beta_y(y-b)], \quad y>y' \qquad (7b)$$

The Wronskian of (14-44c) can now be written as

$$W(y')=\beta_y B_1 D_1\left[\sin(\beta_y y')\cos[\beta_y(y'-b)]-\sin[\beta_y(y'-b)]\cos(\beta_y y')\right]$$

$$W(y')=\beta_y B_1 D_1\left[\sin(\beta_y y'-\beta_y y'+\beta_y b)\right]=\beta_y B_1 D_1\sin(\beta_y b) \qquad (8)$$

Using (14-45a), (14-45b) and (5) with $p(y')=1$, we can write

$$g_m(y;x',y')=\begin{cases}\frac{2}{a\beta_y}\frac{\sin[\beta_y(y'-b)]}{\sin(\beta_y b)}\sin(\beta_y y)\sin\left(\frac{m\pi}{a}x'\right), & y<y' \quad (9a)\\[2ex] \frac{2}{a\beta_y}\frac{\sin(\beta_y y')}{\sin(\beta_y b)}\sin[\beta_y(y-b)]\sin\left(\frac{m\pi}{a}x'\right), & y>y' \quad (9b)\end{cases}$$

Thus the Green's function of (3) can now be expressed as

$$G(x,y;x',y')=\begin{cases}\sum_{m=1,2,..}^{\infty}\frac{2}{a\beta_y}\frac{\sin[\beta_y(y'-b)]}{\sin(\beta_y b)}\sin(\beta_y y)\sin\left(\frac{m\pi}{a}x'\right)\sin\left(\frac{m\pi}{a}x\right), & y<y' \quad (10a)\\[2ex] \sum_{m=1,2,..}^{\infty}\frac{2}{a\beta_y}\frac{\sin(\beta_y y')}{\sin(\beta_y b)}\sin[\beta_y(y-b)]\sin\left(\frac{m\pi}{a}x'\right)\sin\left(\frac{m\pi}{a}x\right), & y>y' \quad (10b)\end{cases}$$

$\boxed{14.12}$ The Green's function for this problem must satisfy the partial differential equation that H_ϕ must satisfy; that is

$$\nabla^2 G + \beta_0^2 G = \delta(\underline{\rho} - \underline{\rho}') \qquad (1)$$

The boundary conditions for the Green's function are chosen to be the same as those for H_ϕ; that is

$$G(\rho=a, 0\le\phi\le 2\pi, 0\le z\le h) = G(\rho=b, 0\le\phi\le 2\pi, 0\le z\le h) = 0 \qquad (2)$$

Let

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{+\infty} g_m(\rho;\rho',\phi')\, e^{jm\phi} \qquad (3)$$

Choosing (14-152d) to represent $\delta(\underline{\rho}-\underline{\rho}')$, we can write (1) in expanded form, assuming no z variations, as (14-153).

Following the procedure outlined on page 898-899, we can write the two solutions for $g_m(\rho;\rho',\phi')$ as given by (14-159a) and (14-159b), or

$$g_m^{(1)} = A_m J_m(\beta_0\rho) + B_m Y_m(\beta_0\rho) \qquad \text{for } \rho<\rho' \qquad (3a)$$

$$g_m^{(2)} = C_m J_m(\beta_0\rho) + D_m Y_m(\beta_0\rho) \qquad \text{for } \rho>\rho' \qquad (3b)$$

The Green's function of (3) must satisfy the boundary conditions of (2). Thus

$$g_m^{(1)}(\rho=a) = A_m J_m(\beta_0 a) + B_m Y_m(\beta_0 a) = 0 \Rightarrow B_m = -A_m \frac{J_m(\beta_0 a)}{Y_m(\beta_0 a)} \qquad (4a)$$

$$g_m^{(2)}(\rho=b) = C_m J_m(\beta_0 b) + D_m Y_m(\beta_0 b) = 0 \Rightarrow D_m = -C_m \frac{J_m(\beta_0 b)}{Y_m(\beta_0 b)} \qquad (4b)$$

Thus (3a) and (3b) reduce to

$$g_m^{(1)} = A_m\left[J_m(\beta_0\rho) - \frac{J_m(\beta_0 a)}{Y_m(\beta_0 a)} Y_m(\beta_0\rho)\right] = A_m\left[J_m(\beta_0\rho) - \alpha Y_m(\beta_0\rho)\right] \qquad (5a)$$

$$g_m^{(2)} = C_m\left[J_m(\beta_0\rho) - \frac{J_m(\beta_0 b)}{Y_m(\beta_0 b)} Y_m(\beta_0\rho)\right] = C_m\left[J_m(\beta_0\rho) - \gamma Y_m(\beta_0\rho)\right] \qquad (5b)$$

Using (14-44c) where $y_1 = g_m^{(1)}$ and $y_2 = g_m^{(2)}$, we can write the Wronskian as

$$W(\rho') = \beta_0 A_m\left[J_m(\beta_0\rho') - \alpha Y_m(\beta_0\rho')\right] C_m\left[J_m'(\beta_0\rho') - \gamma Y_m'(\beta_0\rho)\right]$$
$$- \beta_0 C_m\left[J_m(\beta_0\rho') - \gamma Y_m(\beta_0\rho')\right] A_m\left[J_m'(\beta_0\rho') - \alpha Y_m'(\beta_0\rho)\right]$$

$$W(\rho') = \beta_0 A_m C_m\left[J_m(\beta_0\rho')J_m'(\beta_0\rho') + \alpha\gamma Y_m(\beta_0\rho')Y_m'(\beta_0\rho) - \alpha Y_m(\beta_0\rho')J_m'(\beta_0\rho') - \gamma J_m(\beta_0\rho')Y_m'(\beta_0\rho')\right]$$
$$- \beta_0 C_m A_m\left[J_m(\beta_0\rho')J_m'(\beta_0\rho') + \alpha\gamma Y_m(\beta_0\rho')Y_m'(\beta_0\rho) - \gamma Y_m(\beta_0\rho')J_m'(\beta_0\rho') - \alpha J_m(\beta_0\rho')Y_m'(\beta_0\rho')\right]$$

$$W(\rho') = \beta_0 A_m C_m \left\{ -\alpha\left[J_m'Y_m - J_m Y_m'\right] + \gamma\left[J_m'Y_m - J_m Y_m'\right]\right\}, \quad ' = \frac{\partial}{\partial(\beta_0\rho)} \quad (6)$$

By using the Wronskian for Bessel functions of (11-95), (6) reduces to

$$W(\rho') = +\frac{2}{\pi} A_m C_m \frac{1}{\rho'}\left\{+\alpha - \gamma\right\} = \frac{2}{\pi\rho'} A_m C_m \left[\frac{J_m(\beta_0 a)}{Y_m(\beta_0 a)} - \frac{J_m(\beta_0 b)}{Y_m(\beta_0 b)}\right]$$

$$W(\rho') = \frac{2}{\pi}\frac{1}{\rho'} A_m C_m \frac{J_m(\beta_0 a)Y_m(\beta_0 b) - J_m(\beta_0 b)Y_m(\beta_0 a)}{Y_m(\beta_0 a)Y_m(\beta_0 b)} = \frac{2}{\pi}\frac{A_m C_m}{\rho'}(JY)_m \quad (7)$$

$$\text{where} \quad (JY)_m = \frac{J_m(\beta_0 a)Y_m(\beta_0 b) - J_m(\beta_0 b)Y_m(\beta_0 a)}{Y_m(\beta_0 a)Y_m(\beta_0 b)} \quad (7a)$$

Finally $g_m(\rho;\rho',\phi')$ of (14-156) can be written using (14-158), (5a)-(7a) by referring to (14-45a) and (14-45b), as

$$g_m(\rho;\rho',\phi') = \begin{cases} \frac{1}{4}\left[\frac{J_m(\beta_0\rho') - \gamma Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \alpha Y_m(\beta_0\rho)\right] e^{-jm\phi'}, & a \le \rho \le \rho' \quad (8a) \\ \frac{1}{4}\left[\frac{J_m(\beta_0\rho') - \alpha Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \gamma Y_m(\beta_0\rho)\right] e^{-jm\phi'}, & \rho' \le \rho \le b \quad (8b) \end{cases}$$

Thus the Green's function of (3) can be written as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} \frac{1}{4}\sum_{m=-\infty}^{+\infty}\left[\frac{J_m(\beta_0\rho') - \gamma Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \alpha Y_m(\beta_0\rho)\right] e^{jm(\phi-\phi')} & (9a) \\ \frac{1}{4}\sum_{m=-\infty}^{+\infty}\left[\frac{J_m(\beta_0\rho') - \alpha Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \gamma Y_m(\beta_0\rho)\right] e^{jm(\phi-\phi')} & (9b) \end{cases}$$

14.13 The solution to this problem follows that of Problem 14.12. One of differences are the boundary conditions which for this problem are

$$G(\rho=a, 0\le\phi\le\phi_0, 0\le z\le h) = G(\rho=b, 0\le\phi\le\phi_0, 0\le z\le h) = 0 \quad (1a)$$

$$G(a\le\rho\le b, \phi=0, 0\le z\le h) = G(a\le\rho\le b, \phi=\phi_0, 0\le z\le h) = 0 \quad (1b)$$

To meet the boundary conditions on ϕ, we select a solution for the Green's function of the form

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1,2,}^{+\infty} g_m(\rho;\rho',\phi')\sin\left(\frac{m\pi}{\phi_0}\phi\right) = \sum_{m=1,2,\ldots}^{\infty} g_m(\rho;\rho',\phi')\sin(r\phi), \quad r = \frac{m\pi}{\phi_0} \quad (2)$$

$$m = 1, 2, 3, \ldots$$

Substituting (2) into (14-153) leads to

$$\sum_{m=1,2,\ldots}^{\infty}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{r^2}{\rho^2}+\beta_0^2\right]g_m(\rho;\rho',\phi')\sin(r\phi)=\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \qquad (3)$$

Multiplying both sides of (3) by $\sin(s\phi)$, integrating from 0 to ϕ_0, and using the orthogonality conditions of (14-48a) and (14-48b) leads to

$$\rho\frac{d^2g_m}{d\rho^2}+\frac{dg_m}{d\rho}+\left(\rho\beta_0^2-\frac{r^2}{\rho}\right)g_m=\frac{2}{\phi_0}\sin(r\phi')\delta(\rho-\rho') \qquad (4)$$

the homogeneous form of (4) is

$$\rho\frac{d^2g_m}{d\rho^2}+\frac{dg_m}{d\rho}+\left(\rho\beta_0^2-\frac{r^2}{\rho}\right)g_m=0 \qquad (5)$$

Thus its two solutions are

$$g_m^{(1)}=A_mJ_m(\beta_0\rho)+B_mY_m(\beta_0\rho) \qquad \text{for } \rho<\rho' \qquad (6a)$$

$$g_m^{(2)}=C_mJ_m(\beta_0\rho)+D_mY_m(\beta_0\rho) \qquad \text{for } \rho>\rho' \qquad (6b)$$

Since the boundary conditions in ρ for this problem are the same as those of Problem 14-12, thus we can write using (6)-(8b), (4) and (6a)-(6b) the solution for g_m as

$$g_m(\rho;\rho',\phi')=\begin{cases}\dfrac{\pi}{\phi_0}\left[\dfrac{J_r(\beta_0\rho')-\gamma Y_r(\beta_0\rho')}{(JY)_r}\right]\left[J_r(\beta_0\rho)-\alpha Y_r(\beta_0\rho)\right]\sin(r\phi') & a\le\rho\le\rho' \quad (7a)\\[2ex] \dfrac{\pi}{\phi_0}\left[\dfrac{J_r(\beta_0\rho')-\alpha Y_r(\beta_0\rho')}{(JY)_r}\right]\left[J_r(\beta_0\rho)-\gamma Y_r(\beta_0\rho)\right]\sin(r\phi') & \rho'\le\rho\le b \quad (7b)\end{cases}$$

Cont'd.

$\boxed{14.13 \text{ cont'd.}}$ where

$$\alpha = \frac{J_r(\beta_o a)}{Y_r(\beta_o a)} \qquad (8a)$$

$$\gamma = \frac{J_r(\beta_o b)}{Y_r(\beta_o b)} \qquad (8b)$$

$$(JY)_r = \frac{J_r(\beta_o a)Y_r(\beta_o b) - J_r(\beta_o b)Y_r(\beta_o a)}{Y_r(\beta_o a)\,Y_r(\beta_o b)} \qquad (8c)$$

Thus the Green's function of (2) can be written as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} \dfrac{\pi}{\phi_o}\displaystyle\sum_{m=1,2,\ldots}^{\infty} \left\{\frac{J_{m\pi/\phi_o}(\beta_o\rho') - \gamma Y_{m\pi/\phi_o}(\beta_o\rho')}{(JY)_r}\right\}\left[J_{\frac{m\pi}{\phi_o}}(\beta_o\rho) - \alpha Y_{\frac{m\pi}{\phi_o}}(\beta_o\rho)\right]\sin\left(\frac{m\pi}{\phi_o}\phi'\right)\sin\left(\frac{m\pi}{\phi_o}\phi\right) & a \le \rho \le \rho' \quad (9a) \\[2ex] \dfrac{\pi}{\phi_o}\displaystyle\sum_{m=1,2,\ldots}^{\infty} \left\{\frac{J_{m\pi/\phi_o}(\beta_o\rho') - \alpha Y_{m\pi/\phi_o}(\beta_o\rho')}{(JY)_r}\right\}\left[J_{\frac{m\pi}{\phi_o}}(\beta_o\rho) - \gamma Y_{\frac{m\pi}{\phi_o}}(\beta_o\rho)\right]\sin\left(\frac{m\pi}{\phi_o}\phi'\right)\sin\left(\frac{m\pi}{\phi_o}\phi\right) & \rho' \le \rho \le b \quad (9b) \end{cases}$$

$\boxed{14.15}$ The solution to this problem proceeds in the same way until we reach (14-156), or

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{+\infty} g_m(\rho;\rho',\phi')e^{jm\phi} \qquad (1)$$

$$\rho\frac{d^2g_m}{d\rho^2} + \frac{dg_m}{d\rho} + \left(\rho\beta_o^2 - \frac{m^2}{\rho}\right)g_m = \frac{e^{-jm\phi'}}{2\pi}\delta(\rho-\rho') \qquad (2)$$

Now instead of using a closed form for the solution of $g_m(\rho;\rho',\phi')$ of (2), as outlined in (14-52)-(14-61), we will select a series form. We write the desired series solution as

$$g_m(\rho;\rho',\phi') = \sum_{n=1,2,\ldots}^{\infty}\left[A_{mn}J_m(\beta_{mn}\rho) + B_{mn}Y_m(\beta_{mn}\rho)\right] \qquad (3)$$

Since the Green's function must be finite everywhere, including $\rho=0$, then $g_m(\rho;\rho',\phi')$ must also be finite everywhere (including $\rho=0$). Thus $B_{mn}=0$. Therefore

$$g_m(\rho;\rho',\phi') = \sum_{n=1,2,..}^{\infty} A_{mn} J_m(\beta_{mn}\rho) \quad \text{of the form of (14-56)} \qquad (4)$$

Now we apply the boundary condition of (14-149a) which leads to

$$g_m(\rho=a;\rho',\phi') = \sum_{n=1,2,..}^{\infty} A_{mn}(\rho',\phi') J_m(\beta_{mn}a) = 0 \Rightarrow \beta_{mn} = \chi_{mn} = \text{roots of Bessel function } J_m$$

Thus

$$\beta_{mn} = \left(\frac{\chi_{mn}}{a}\right), \quad \begin{matrix} m = 0, \pm1, \pm2, \ldots \\ n = 1, 2, 3, \ldots \end{matrix} \qquad (5)$$

$$g_m(\rho;\rho',\phi') = \sum_{n=1,2,..}^{\infty} A_{mn}(\rho',\phi') J_m(\beta_{mn}\rho), \quad \beta_{mn} = \frac{\chi_{mn}}{a} \qquad (6)$$

The solution for $g_m(\rho;\rho',\phi')$ as given by (3) or (4) was chosen to satisfy the homogeneous differential equation of

$$\rho\frac{d^2 J_m}{d\rho^2} + \frac{dJ_m}{d\rho} + \left(\rho\beta_{mn}^2 - \frac{m^2}{\rho}\right)J_m = 0 \qquad (7)$$

where $\psi_m(\rho)$ in (14-56) is $\psi_m(\rho) = J_m(\beta_{mn}\rho)$ of the form of (14-54)

Substituting (6) in (2) we can write that

$$\sum_{n=1,2,..}^{\infty}\left\{\rho\frac{d^2 J_m}{d\rho^2} + \frac{dJ_m}{d\rho} + \left(\rho\beta_{mn}^2 - \frac{m^2}{\rho}\right)J_m + \left(\rho\beta_0^2 - \rho\beta_{mn}^2\right)J_m\right\}A_{mn} = \frac{e^{-jm\phi'}}{2\pi}\delta(\rho-\rho') \qquad (8)$$

Substituting (7) in (8) leads to

$$\sum_{n=1,2,..}^{\infty} A_{mn}(\beta_0^2 - \beta_{mn}^2)\rho J_m(\beta_{mn}\rho) = \frac{e^{-jm\phi'}}{2\pi}\delta(\rho-\rho') \qquad (9)$$

Since the eigenfunction $J_{mn}(\beta_{mn}\rho)$ are orthogonal over the interval 0 to a for the same value of m but different value of n (with the

$\boxed{\text{14.15 cont'd.}}$ weighting function ρ) as [1], [2]

$$\int_0^a \rho\, J_m(\beta_{mn}\rho) J_m(\beta_{mp}\rho)\, d\rho = \delta_{np} N \quad \text{where } \delta_{np} = \begin{cases} 1 & p = n \\ 0 & p \neq n \end{cases} \qquad (10)$$

then we multiply both sides of (9) by $J_m(\beta_{mp}\rho)$ and integrate from 0 to a. Doing this and using that

$$\int_0^a \rho\, J_m(\beta_{mn}\rho) J_m(\beta_{mp}\rho)\, d\rho = \begin{cases} \frac{a^2}{2} J_{m+1}^2(\beta_{mn}a) & p = n \quad (11a) \\ 0 & p \neq n \quad (11b) \end{cases}$$

we can write that

$$A_{mn}(\beta_0^2 - \beta_{mn}^2)\frac{a^2}{2} J_{m+1}^2(\beta_{mn}a) = \frac{e^{-jm\phi'}}{2\pi} J_m(\beta_{mn}\rho') \qquad (12)$$

or

$$A_{mn} = \frac{1}{\pi a^2(\beta_0^2 - \beta_{mn}^2) J_{m+1}^2(\beta_{mn}a)} e^{-jm\phi'} J_m(\beta_{mn}\rho') \qquad (12a)$$

Therefore $g_m(\rho; \rho', \phi')$ of (6) can be written as

$$g_m(\rho; \rho', \phi') = \sum_{n=1,2,\ldots}^{\infty} \frac{1}{\pi a^2(\beta_0^2 - \beta_{mn}^2) J_{m+1}^2(\beta_{mn}a)} e^{-jm\phi'} J_m(\beta_{mn}\rho') J_m(\beta_{mn}\rho) \qquad (13)$$

Ultimately the Green's function $(\rho, \phi; \rho', \phi')$ of (14-151) can be written as

$$G(\rho, \phi; \rho', \phi') = \frac{1}{\pi a^2} \sum_{m=-\infty}^{+\infty} \left[\sum_{n=1}^{\infty} \frac{1}{(\beta_0^2 - \beta_{mn}^2) J_{m+1}^2(\beta_{mn}a)} J_m(\beta_{mn}\rho') J_m(\beta_{mn}\rho) \right] e^{jm(\phi-\phi')} \qquad (14)$$

$\boxed{14.20}$ To solve this problem, it is easier to assume that the point source of Figure P14.20 is located at the origin. When that is done, the

(a). field radiated by the point source is only a function of r (not a function of θ or ϕ). Thus in expanded form (14-166) can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)+\beta^2 G = \frac{\partial^2 G}{\partial r^2}+\frac{2}{r}\frac{\partial G}{\partial r}+\beta^2 G = \frac{1}{4\pi r^2}\delta(r-r') \qquad (1)$$

$$\text{with} \quad G(r=\infty) = \text{finite} \qquad (1a)$$

Since $G(r,r')$ is only a function of r, (1) can be written as

$$\frac{d^2G}{dr^2}+\frac{2}{r}\frac{dG}{dr}+\beta^2 G = \frac{1}{4\pi r^2}\delta(r-r') \Rightarrow r^2\frac{d^2G}{dr^2}+2r\frac{dG}{dr}+(\beta r)^2 G = \frac{\delta(r-r')}{4\pi} \qquad (2)$$

A solution of the homogeneous form of (2) takes the form of

$$G(r) = A\frac{e^{-j\beta r}}{r}+B\frac{e^{+j\beta r}}{r} \qquad (3)$$

For an $e^{j\omega t}$ time convention, the boundary condition of (1a) allows us to write that $B=0$. Thus (3) reduces to

$$G(r) = A\frac{e^{-j\beta r}}{r} \qquad (4)$$

To evaluate the coefficient A in (4), we can integrate (2) over an infinitesimal volume which surrounds the origin and to then allow the radial distance r to approach zero $(r\to 0)$. Doing this we find that $A=-\frac{1}{4\pi}$ which allows us to write (4) as

$$G(r) = -\frac{e^{-j\beta r}}{4\pi r} \qquad (5)$$

When the source is removed from the origin, as shown in Figure P14.20, the Green's function of (5) can be written as

$$G = -\frac{1}{4\pi}\frac{e^{-j\beta R}}{R} = -\frac{1}{4\pi}\frac{e^{-j\beta|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \qquad (6)$$

(b). The reduction of the three-dimensional Green's function of (6) to that of the two-dimensional one of Example is obtained by the use of the integral of (11-28a). This allows us to reduce the three-dimensional problem to a two-dimensional one.