A thin linear electric current element of very short length ($l \ll \lambda$) and with constant current

$$\bar{I}_e(z') = \hat{a}_z I_e$$

is positioned symmetrically at the origin and oriented along the z axis. Determine Electric and magnetic fields radiated by the dipole

<u>Soln</u>

$$\bar{A}(x,y,z) = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \hat{a}_z I_e \frac{e^{-j\beta R}}{R} dz'$$

As $l \to 0$ ($l \ll \lambda$) then $R = r$

$$\bar{A}(x,y,z) = \frac{\mu I_e}{4\pi} \int_{-l/2}^{l/2} \hat{a}_z \frac{e^{-j\beta r}}{r} dz'$$

$$\bar{A}(x,y,z) = \hat{a}_z \frac{\mu}{4\pi} \frac{e^{-j\beta r}}{r} I_e l$$

Transforming vector potential $\bar{A}$ from rectangular to circular (spherical) components we can write

$$\bar{A}_r = \bar{A}_z \cos\theta = \frac{\mu I_e l e^{-j\beta r}}{4\pi r} \cos\theta$$

$$\bar{A}_\theta = -\bar{A}_z \sin\theta = -\frac{\mu I_e l e^{-j\beta r}}{4\pi r} \sin\theta \qquad \bar{A}_\phi = 0$$

From (6.32) we get

$$\bar{H} = \frac{1}{\mu}(\nabla \times \bar{A})$$

$$\nabla \times \bar{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]$$

$$+ \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\hat{a}_r}{r\sin\theta}\left(-\frac{\partial}{\partial\phi}\cancelto{0}{\left(-\frac{\mu I_e l e^{-j\beta r}}{4\pi r}\sin\theta\right)}\right) + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}\cancelto{0}{r \times 0}\right]$$

$$+ \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}\left[r\left(-\frac{\mu I_e l e^{-j\beta r}}{4\pi r}\sin\theta\right)\right] - \frac{\partial}{\partial\theta}\left[\frac{\mu I_e l e^{-j\beta r}}{4\pi r}\cos\theta\right]\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[-\frac{\mu I_e l}{4\pi}\left[\frac{\partial}{\partial r}e^{-j\beta r}\sin\theta\right] - \frac{\mu I_e l e^{-j\beta r}}{4\pi r}\frac{\partial}{\partial\theta}\cos\theta\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[-\frac{\mu I_e l}{4\pi}\sin\theta(-j\beta)e^{-j\beta r} - \frac{\mu I_e l e^{-j\beta r}}{4\pi r}(-\sin\theta)\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[\frac{\mu I_e l}{4\pi} j\beta \sin\theta e^{-j\beta r} + \frac{\mu I_e l e^{-j\beta r}}{4\pi r}\sin\theta\right]$$

$$\nabla \times \bar{A} = \hat{a}_\phi \frac{\mu I_e l}{4\pi r} j\beta \sin\theta \, e^{-j\beta r}\left[1 + \frac{1}{j\beta r}\right]$$

$$\bar{H} = \frac{1}{\mu}(\nabla \times \bar{A})$$

$$\bar{H} = \hat{a}_\phi \, j\frac{\beta I_e l}{4\pi r} \sin\theta \left[1 + \frac{1}{j\beta r}\right] e^{-j\beta r}$$

$$\bar{H}_\phi = j\frac{\beta I_e l}{4\pi r} \sin\theta \left[1 + \frac{1}{j\beta r}\right] e^{-j\beta r}$$

The electric field $\bar{E}$ is given as

$$\bar{E} = -j\omega A - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla \cdot \bar{A}) = \frac{1}{j\omega\varepsilon}\nabla \times \bar{H}$$

and leads to

$$E_r = \eta \frac{I_e l \cos\theta}{2\pi r^2}\left[1 + \frac{1}{j\beta r}\right] e^{-j\beta r}$$

$$E_\theta = j\eta \frac{\beta I_e l \sin\theta}{4\pi r}\left[1 + \frac{1}{j\beta r} - \frac{1}{(\beta r)^2}\right] e^{-j\beta r}$$

FAR FIELD RADIATION

The fields radiated by antennas of finite dimensions are spherical waves. The general solution of the vector in spherical coordinates can be given as:

$$\bar{A} = \hat{a}_r A_r(r,\theta,\phi) + \hat{a}_\theta A_\theta(r,\theta,\phi) + \hat{a}_\phi A_\phi(r,\theta,\phi)$$

$$\bar{A} = [\hat{a}_r A_r'(\theta,\phi) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)] \frac{e^{-j\beta r}}{r}$$

The r variations are separable from those of θ and ϕ.

Substituting in eqn:-

$$\bar{E} = -j\omega\bar{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla\cdot A)$$

we get

$$\bar{E} = \frac{1}{r}\left\{-j\omega e^{-j\beta r}[\hat{a}_r(0) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)]\right.$$

$$+ \frac{1}{r^2}\{---\} + \{---\}$$

The radial E-field component has no $\frac{1}{r}$ terms because its contributions from the first and second terms of (6.17) cancel each other.

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$$\bar{H} = \frac{1}{r}\left\{-j\frac{\omega}{\eta} e^{-j\beta r}[\hat{a}_r(0) + \hat{a}_\theta A_\phi'(\theta,\phi) - \hat{a}_\phi A_\theta'(\theta,\phi)]\right\}$$

$$+\frac{1}{r}\{---\} + ---$$

where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is intrinsic impedance of the medium

Neglecting higher order terms of $1/r^n$, the radiated $\bar{E}$ and $\bar{H}$ fields have only θ and ϕ components. They can be expressed as.

Far-Field Region

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \quad \bar{E}_A \simeq -j\omega \bar{A}$$

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq j\frac{\omega}{\eta} A_\phi = -\frac{E_\phi}{\eta} \\ H_\phi &\simeq -j\frac{\omega}{\eta} A_\theta = \frac{E_\theta}{\eta} \end{aligned}\right\} \quad \bar{H}_A \simeq \frac{\hat{a}_r}{\eta} \times \bar{E}_A = -j\frac{\omega}{\eta}\hat{a}_r \times A$$

$|||^{ly}$ the far-zone fields that are due to a magnetic source $\bar{M}$ can be written as.

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &= -j\omega F_\theta \\ H_\phi &\simeq -j\omega F_\phi \end{aligned}\right\} \bar{H}_F \simeq -j\omega \bar{F} \qquad \left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega\eta F_\phi = \eta H_\phi \\ E_\phi &\simeq j\omega\eta F_\theta = -\eta H_\theta \end{aligned}\right\} \begin{aligned} \bar{E}_F &\simeq -\eta\,\hat{a}_r \times \bar{H}_F \\ &= j\omega\eta\,\hat{a}_r \times \bar{F} \end{aligned}$$

Radiation and Scattering Equations

Near Field

Vector potential $\bar{A}$ that is due to current density $\bar{J}$ is given by:

$$\bar{A}(x,y,z) = \frac{\mu}{4\pi} \iiint_{v} \bar{J}(x',y',z') \frac{e^{-j\beta R}}{R} \, dv'$$

where primed coordinates (x',y',z') represent the source and unprimed coordinates (x,y,z) represent observation point.

The magnetic field due to the potential is given by

$$\bar{H}_A = \frac{1}{\mu} \nabla \times \bar{A} = \frac{1}{4\pi} \nabla \times \iiint_{V} \bar{J}(x',y',z') \frac{e^{-j\beta R}}{R} \, dv'$$

Interchanging integration and differentiation, we get

$$\bar{H}_A = \frac{1}{4\pi} \iiint_{v} \nabla \times \left[\bar{J}(x',y',z') \frac{e^{-j\beta R}}{R} \right] dv'$$

Using vector identity

$$\nabla \times (g\bar{F}) = (\nabla g) \times \bar{F} + g(\nabla \times \bar{F})$$

we can write

$$\nabla \times \left[\frac{e^{-j\beta R}}{R} \bar{J}(x',y',z') \right] = \nabla\left(\frac{e^{-j\beta R}}{R} \right) \times \bar{J}(x',y',z') + \frac{e^{-j\beta R}}{R} \nabla \times \bar{J}(x',y',z')$$

Since $\bar{J}$ is function of primed coordinates & $\bar{\nabla}$ is a function of unprimed coordinates

$$\nabla \times \bar{J}(x', y', z') = 0$$

also

$$\nabla\left(\frac{e^{-j\beta R}}{R}\right) = -\hat{R}\left(\frac{1+j\beta R}{R^2}\right)e^{-j\beta R}$$

where $\hat{R}$ is a unit vector directed along the line joining any point of the source and the observation point.

$$\bar{H}_A(x, y, z) = \frac{-1}{4\pi}\iiint_v (\hat{R}\times\bar{J})\ \frac{1+j\beta R}{R^2}\ e^{-j\beta R}\, dx'dy'dz'$$

which can be expanded in its rectangular components as.

$$\bar{H}_{Ax} = \frac{1}{4\pi}\iiint_v [(z-z')\bar{J}_y - (y-y')\bar{J}_z]\ \frac{1+j\beta R}{R^3}\ e^{-j\beta R}\, dx'dy'dz'$$

$$\bar{H}_{Ay} = \frac{1}{4\pi}\iiint_v [(x-x')\bar{J}_z - (z-z')\bar{J}_x]\ \frac{1+j\beta R}{R^3}\ e^{-j\beta R}\, dx'dy'dz'$$

$$\bar{H}_{Az} = \frac{1}{4\pi}\iiint_v [(y-y')\bar{J}_x - (x-x')\bar{J}_y]\ \frac{1+j\beta R}{R^3}\ e^{-j\beta R}\, dx'dy'dz'$$

And electric field can be written as

$$\bar{E}_A = \hat{a}_x E_{Ax} + \hat{a}_y E_{ay} + \hat{a}_z E_{az} = -j\omega A - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{A}) = \frac{1}{j\omega\varepsilon}\nabla\times \bar{H}_A$$

$$\nabla\left(\frac{e^{-j\beta R}}{R}\right] = \hat{R}\,\frac{d}{dR}\left[\frac{e^{-j\beta R}}{R}\right]$$

$$= \hat{R}\left[\frac{1}{R}\frac{d}{dR}e^{-j\beta R} + e^{-j\beta R}\frac{d}{dR}\left[\frac{1}{R}\right]\right]$$

$$= \hat{R}\ \frac{1}{R}(-j\beta)\,e^{-j\beta R} + e^{-j\beta R}(-1)R^{-2}\hat{R}$$

$$= -\hat{R}\left[\frac{j\beta}{R}e^{-j\beta R} + \frac{e^{-j\beta R}}{R^2}\right]$$

$$\bar{\nabla}\left[\frac{e^{-j\beta R}}{R}\right] = -\hat{R}\left[\frac{1+j\beta R}{R^2}\right]e^{-j\beta R}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ (x-x') & (y-y') & (z-z') \\ \bar{J}_x & \bar{J}_y & \bar{J}_z \end{vmatrix} \quad \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \to \frac{1}{R}$$

$$\hat{x}\left[(y-y')\bar{J}_z - (z-z')\bar{J}_y\right] - \hat{y}\left[(x-x')\bar{J}_z - (z-z')\bar{J}_x\right]$$
$$+\hat{z}\left[(x-x')\bar{J}_y - (y-y')\bar{J}_x\right]$$

$$\bar{E}_{Ax} = \frac{-j\eta}{4\pi\beta} \iiint_v \left(G_1 \bar{J}_x + (x-x') G_2 \times \left[(x-x') \bar{J}_x + (y-y') \bar{J}_y + (z-z') \bar{J}_z \right] \right) e^{-j\beta R} dx' dy' dz'$$

$$\bar{E}_{Ay} = \frac{-j\eta}{4\pi\beta} \iiint_v \left(G_1 \bar{J}_y + (y-y') G_2 \times \left[(x-x') \bar{J}_x + (y-y') \bar{J}_y + (z-z') \bar{J}_z \right] \right) e^{-j\beta R} dx' dy' dz'$$

$$\bar{E}_{Az} = \frac{-j\eta}{4\pi\beta} \iiint_v \left(G_1 \bar{J}_z + (z-z') G_2 \times \left[(x-x') \bar{J}_x + (y-y') \bar{J}_y + (z-z') \bar{J}_z \right] \right) e^{-j\beta R} dx' dy' dz'$$

where

$$\bar{G}_1 = \frac{-1 - j\beta R + \beta^2 R^2}{R^3}$$

$$\bar{G}_2 = \frac{3 + j3\beta R - \beta^2 R^2}{R^5}$$

We can write the electric vector potential as

$$\bar{F}(x,y,z) = \frac{\varepsilon}{4\pi} \iiint_v \bar{M}(x',y',z') \frac{e^{-j\beta R}}{R} dv'$$

The electric field component

$$\bar{E}_F = -\frac{1}{\varepsilon} \nabla \times \bar{F}$$

as

$$\bar{E}_{F_x} = \frac{-1}{4\pi} \iiint_v [(z-z')\bar{m}_y - (y-y')\bar{m}_z] \frac{1+j\beta R}{R^3} e^{-j\beta R} dx'dy'dz'$$

$$\bar{E}_{F_y} = \frac{-1}{4\pi} \iiint_v [(x-x')\bar{m}_z - (z-z')\bar{m}_x] \frac{1+j\beta R}{R^3} e^{-j\beta R} dx'dy'dz'$$

$$\bar{E}_{F_z} = \frac{-1}{4\pi} \iiint_v [(y-y')\bar{m}_x - (x-x')\bar{m}_y] \frac{1+j\beta R}{R^3} e^{-j\beta R} dx'dy'dz'$$

&

$$\bar{H}_F = -j\omega\bar{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla\cdot\bar{F}) = -\frac{1}{j\omega\mu} \nabla\times\bar{E}_F$$

$$\bar{H}_{F_x} = \frac{-j}{4\pi\beta\eta} \iiint_v \{G_1\bar{m}_x + (x-x')G_2 \times [(x-x')\bar{m}_x + (y-y')\bar{m}_y + (z-z')\bar{m}_z]\} e^{-j\beta R} dx'dy'dz'$$

$$\bar{H}_{F_y} = \frac{-j}{4\pi\beta\eta} \iiint_v \{G_1\bar{m}_y + (y-y')G_2 \times [(x-x')\bar{m}_x + (y-y')\bar{m}_y + (z-z')\bar{m}_z]\} e^{-j\beta R} dx'dy'dz'$$

$$\bar{H}_{F_z} = \frac{-j}{4\pi\beta\eta} \iiint_v \{G_1\bar{m}_z + (z-z')G_2 \times [(x-x')\bar{m}_x + (y-y')\bar{m}_y + (z-z')\bar{m}_z]\} e^{-j\beta R} dx'dy'dz'$$

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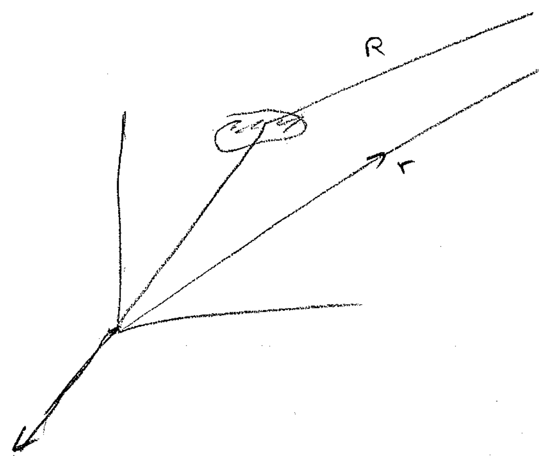
A thin linear electric current element of very short length ($l \ll \lambda$) and with constant current

$$\bar{I}_e(z') = \hat{a}_z I_e$$

is positioned symmetrically at the origin and oriented along the z axis. Determine Electric and magnetic fields radiated by the dipole

<u>Soln</u>



$$\bar{A}(x,y,z) = \frac{\mu}{4\pi}\int_{-l/2}^{l/2} \hat{a}_z I_e \frac{e^{-j\beta R}}{R}\, dz'$$

As $l \to 0$ ($l \ll \lambda$) then $R = r$

$$\bar{A}(x,y,z) = \frac{\mu I_e}{4\pi}\int_{-l/2}^{l/2} \hat{a}_z \frac{e^{-j\beta r}}{r}\, dz'$$

$$\bar{A}(x,y,z) = \hat{a}_z \frac{\mu}{4\pi}\frac{e^{-j\beta r}}{r} I_e l$$

$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$

$A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$

$A_\phi = -A_x \sin\phi + A_y \cos\phi$

Transforming vector potential $\bar{A}$ from rectangular to circular (spherical) components we can write

$$\bar{A}_r = \bar{A}_z \cos\theta = \frac{\mu I_e l e^{-j\beta r}}{4\pi r}\cos\theta$$

$$\bar{A}_\phi = 0$$

$$\bar{A}_\theta = -\bar{A}_z \sin\theta = -\frac{\mu I_e l e^{-j\beta r}}{4\pi r}\sin\theta$$

From (6-32) we get

$$\bar{H} = \frac{1}{\mu}(\nabla \times \bar{A})$$

$$\nabla \times A = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]$$

$$+ \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\hat{a}_r}{r\sin\theta}\left(-\frac{\partial}{\partial\phi}\left(\cancelto{0}{-\frac{\mu I_e l e^{-j\beta r}}{4\pi r}}\sin\theta\right)\right) + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}\cancelto{0}{r} \times 0\right]$$

$$+ \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}\left[r\left(-\frac{\mu I_e l e^{-j\beta r}}{4\pi r}\sin\theta\right)\right] - \frac{\partial}{\partial\theta}\left[\frac{\mu I_e l e^{-j\beta r}}{4\pi r}\cos\theta\right]\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[-\frac{\mu I_e l}{}\left[\frac{\partial}{\partial r}e^{-j\beta r}\sin\theta\right] - \frac{\mu I_e l e^{-j\beta r}}{4\pi r}\frac{\partial}{\partial\theta}\cos\theta\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[-\frac{\mu I_e l}{4\pi}\sin\theta\,(-j\beta)\,e^{-j\beta r} - \frac{\mu I_e l e^{-j\beta r}}{4\pi r}(-\sin\theta)\right]$$

$$= \frac{\hat{a}_\phi}{r}\left[\frac{\mu I_e l}{4\pi}\,j\beta\,\sin\theta\, e^{-j\beta r} + \frac{\mu I_e l\, e^{-j\beta r}}{4\pi r}\sin\theta\right]$$

$$\nabla \times \bar{A} = \hat{a}_\phi \frac{\mu I_e l}{4\pi r} j\beta \sin\theta \, e^{-j\beta r}\left[1 + \frac{1}{j\beta r}\right]$$

$$\bar{H} = \frac{1}{\mu}(\nabla \times \bar{A})$$

$$\bar{H} = \hat{a}_\phi \, j\frac{\beta I_e l}{4\pi r}\sin\theta \left[1 + \frac{1}{j\beta r}\right] e^{-j\beta r}$$

$$\bar{H}_\phi = j\frac{\beta I_e l}{4\pi r}\sin\theta \left[1 + \frac{1}{j\beta r}\right] e^{-j\beta r}$$

The electric field $\bar{E}$ is given as

$$\bar{E} = -j\omega \bar{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{A}) = \frac{1}{j\omega\varepsilon}\nabla \times \bar{H}$$

and leads to

$$E_r = \eta \frac{I_e l \cos\theta}{2\pi r^2}\left[1 + \frac{1}{j\beta r}\right] e^{-j\beta r}$$

$$E_\theta = j\eta \frac{\beta I_e l \sin\theta}{4\pi r}\left[1 + \frac{1}{j\beta r} - \frac{1}{(\beta r)^2}\right] e^{-j\beta r}$$

FAR FIELD RADIATION

The fields radiated by antennas of finite dimensions are spherical waves. The general solution of the vector in spherical coordinates can be given as:

$$\bar{A} = \hat{a}_r A_r(r,\theta,\phi) + \hat{a}_\theta A_\theta(r,\theta,\phi) + \hat{a}_\phi A_\phi(r,\theta,\phi)$$

$$\bar{A} = [\hat{a}_r A_r'(\theta,\phi) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)] \frac{e^{-j\beta r}}{r}$$

The r variations are separable from those of θ and ϕ.

Substituting in eqn:-

$$\bar{E} = -j\omega\bar{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla\cdot A)$$

we get

$$\bar{E} = \frac{1}{r} \{-j\omega e^{-j\beta r} [\hat{a}_r(0) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)]$$

$$+ \frac{1}{r^2} \{---\} + \{---\}$$

The radial E-field component has no $\frac{1}{r}$ terms because its contributions from the first and second terms of (6.17) cancel eachother.

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$$\bar{H} = \frac{1}{r} \left\{ -j\frac{\omega}{\eta} e^{-j\beta r} [\hat{a}_r(0) + \hat{a}_\theta A_\phi'(\theta,\phi) - \hat{a}_\phi A_\theta'(\theta,\phi)] \right\}$$

$$+ \frac{1}{r} \{ — \} + —$$

where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is intrinsic impedance of the medium

Neglecting higher order terms of $1/r^n$, the radiated $\bar{E}$ and $\bar{H}$ fields have only θ and ϕ components. They can be expressed as.

Far-Field Region

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \quad \bar{E}_A = -j\omega \bar{A}$$

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq j\frac{\omega}{\eta} A_\phi = -\frac{E_\phi}{\eta} \\ H_\phi &\simeq -j\frac{\omega}{\eta} A_\theta = \frac{E_\theta}{\eta} \end{aligned}\right\} \quad \bar{H}_A \simeq \frac{\hat{a}_r}{\eta} \times \bar{E}_A = -j\frac{\omega}{\eta} \hat{a}_r \times A$$

Ill'ly the far-zone fields that are due to a magnetic source $\bar{M}$ can be written as.

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &= -j\omega F_\theta \\ H_\phi &= -j\omega F_\phi \end{aligned}\right\} \quad \bar{H}_F = -j\omega \bar{F} \qquad \left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega\eta F_\phi = \eta H_\phi \\ E_\phi &= j\omega\eta F_\theta = -\eta H_\theta \end{aligned}\right\} \quad \begin{aligned} \bar{E}_F &= -\eta \hat{a}_r \times \bar{H}_F \\ &= j\omega\eta \hat{a}_r \times \bar{F} \end{aligned}$$

Radiation and Scattering Equations

Near Field

Vector potential $\bar{A}$ that is due to current density $\bar{J}$ is given by:

$$\bar{A}(x,y,z) = \frac{\mu}{4\pi} \iiint_v \bar{J}(x',y',z') \frac{e^{-j\beta R}}{R} dv'$$

where primed coordinates (x',y',z') represent the source and unprimed coordinates (x,y,z) represent observation point.

The magnetic field due to the potential is given by

$$\bar{H}_A = \frac{1}{\mu} \nabla \times \bar{A} = \frac{1}{4\pi} \nabla \times \iiint_V \bar{J}(x',y',z') \frac{e^{-j\beta R}}{R} dv'$$

Interchanging integration and differentiation, we get

$$\bar{H}_A = \frac{1}{4\pi} \iiint_v \nabla \times \left[\bar{J}(x',y',z') \frac{e^{-j\beta R}}{R} \right] dv'$$

Using vector identity

$$\nabla \times (g\bar{F}) = (\nabla g) \times \bar{F} + g(\nabla \times \bar{F})$$

we can write

$$\nabla \times \left[\frac{e^{-j\beta R}}{R} \bar{J}(x',y',z') \right] = \nabla\left(\frac{e^{-j\beta R}}{R} \right) \times \bar{J}(x',y',z') + \frac{e^{-j\beta R}}{R} \nabla \times \bar{J}(x',y',z')$$

Since $\bar{J}$ is function of primed coordinates & $\bar{\nabla}$ is a function of unprimed coordinates

$$\nabla \times \bar{J}(x', y', z') = 0$$

also

$$\nabla\left(\frac{e^{-j\beta R}}{R}\right) = -\hat{R}\left(\frac{1+j\beta R}{R^2}\right)e^{-j\beta R}$$

where $\hat{R}$ is a unit vector directed along the line joining any point of the source and the observation point.

$$\bar{H}_A(x, y, z) = \frac{-1}{4\pi}\iiint_v (\hat{R}\times\bar{J})\,\frac{1+j\beta R}{R^2}\,e^{-j\beta R}\,dx'dy'dz'$$

which can be expanded in its rectangular components as.

$$\bar{H}_{Ax} = \frac{1}{4\pi}\iiint_v [(z-z')\bar{J}_y - (y-y')\bar{J}_z]\,\frac{1+j\beta R}{R^3}\,e^{-j\beta R}\,dx'dy'dz'$$

$$\bar{H}_{Ay} = \frac{1}{4\pi}\iiint_v [(x-x')\bar{J}_z - (z-z')\bar{J}_x]\,\frac{1+j\beta R}{R^3}\,e^{-j\beta R}\,dx'dy'dz'$$

$$\bar{H}_{Az} = \frac{1}{4\pi}\iiint_v [(y-y')\bar{J}_x - (x-x')\bar{J}_y]\,\frac{1+j\beta R}{R^3}\,e^{-j\beta R}\,dx'dy'dz'$$

And electric field can be written as

$$\bar{E}_A = \hat{a}_x E_{Ax} + \hat{a}_y E_{ay} + \hat{a}_z E_{az} = -j\omega A - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{A}) = \frac{1}{j\omega\varepsilon}\nabla\times\bar{H}_A$$

$$\nabla\left(\frac{e^{-j\beta R}}{R}\right] = \hat{R}\,\frac{d}{dR}\left[\frac{e^{-j\beta R}}{R}\right]$$

$$= \hat{R}\left[\frac{1}{R}\frac{d}{dR}e^{-j\beta R} + e^{-j\beta R}\frac{d}{dR}\left[\frac{1}{R}\right]\right]$$

$$= \hat{R}\quad \frac{1}{R}(-j\beta)\,e^{-j\beta R} + e^{-j\beta R}(-1)R^{-2}\hat{R}$$

$$= -\hat{R}\left[\frac{j\beta}{R}e^{-j\beta R} + \frac{e^{-j\beta R}}{R^2}\right]$$

$$\bar{\nabla}\left[\frac{e^{-j\beta R}}{R}\right] = -\hat{R}\left[\frac{1+j\beta R}{R^2}\right]e^{-j\beta R}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ (x-x') & (y-y') & (z-z') \\ \bar{J}_x & \bar{J}_y & \bar{J}_z \end{vmatrix} \quad \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \rightarrow \frac{1}{R}$$

$$\hat{x}\left[(y-y')\bar{J}_z - (z-z')\bar{J}_y\right] - \hat{y}\left[(x-x')\bar{J}_z - (z-z')\bar{J}_x\right]$$

$$+\hat{z}\left[(x-x')\bar{J}_y - (y-y')\bar{J}_x\right]$$

$$\bar{E}_{Ax} = \frac{-j\eta}{4\pi\beta} \iiint_v \left(G_1 \bar{J}_x + (x-x') G_2 \times \left[(x-x')\bar{J}_x + (y-y')\bar{J}_y + (z-z')\bar{J}_z \right] \right) e^{-j\beta R} dx'dy'dz'$$

$$\bar{E}_{Ay} = \frac{-j\eta}{4\pi\beta} \iiint_v \left(G_1 \bar{J}_y + (y-y') G_2 \times \left[(x-x')\bar{J}_x + (y-y')\bar{J}_y + (z-z')\bar{J}_z \right] \right) e^{-j\beta R} dx'dy'dz'$$

$$\bar{E}_{Az} = \frac{-j\eta}{4\pi\beta} \iiint_v \left(G_1 \bar{J}_z + (z-z') G_2 \times \left[(x-x')\bar{J}_x + (y-y')\bar{J}_y + (z-z')\bar{J}_z \right] \right) e^{-j\beta R} dx'dy'dz'$$

where

$$\bar{G}_1 = \frac{-1-j\beta R + \beta^2 R^2}{R^3}$$

$$\bar{G}_2 = \frac{3+j3\beta R - \beta^2 R^2}{R^5}$$

We can write the electric vector potential as

$$\bar{F}(x,y,z) = \frac{\varepsilon}{4\pi} \iiint_v \bar{M}(x',y',z') \frac{e^{-j\beta R}}{R} dv'$$

The electric field component

$$\bar{E}_F = -\frac{1}{\varepsilon} \nabla \times \bar{F}$$

as

$$\bar{E}_{F_x} = \frac{-1}{4\pi} \iiint_v [(z-z')\bar{M}_y - (y-y')\bar{M}_z] \frac{1+j\beta R}{R^3} e^{-j\beta R} dx'dy'dz'$$

$$\bar{E}_{F_y} = \frac{-1}{4\pi} \iiint_v [(x-x')\bar{M}_z - (z-z')\bar{M}_x] \frac{1+j\beta R}{R^3} e^{-j\beta R} dx'dy'dz'$$

$$\bar{E}_{F_z} = \frac{-1}{4\pi} \iiint_v [(y-y')\bar{M}_x - (x-x')\bar{M}_y] \frac{1+j\beta R}{R^3} e^{-j\beta R} dx'dy'dz'$$

&

$$\bar{H}_F = -j\omega\bar{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla\cdot\bar{F}) = -\frac{1}{j\omega\mu} \nabla\times\bar{E}_F$$

$$\bar{H}_{F_x} = \frac{-j}{4\pi\beta\eta} \iiint_v \{G_1\bar{M}_x + (x-x')G_2 \times [(x-x')\bar{M}_x + (y-y')\bar{M}_y + (z-z')\bar{M}_z]\} e^{-j\beta R} dx'dy'dz'$$

$$\bar{H}_{F_y} = \frac{-j}{4\pi\beta\eta} \iiint_v \{G_1\bar{M}_y + (y-y')G_2 \times [(x-x')\bar{M}_x + (y-y')\bar{M}_y + (z-z')\bar{M}_z]\} e^{-j\beta R} dx'dy'dz'$$

$$\bar{H}_{F_z} = \frac{-j}{4\pi\beta\eta} \iiint_v \{G_1\bar{M}_z + (z-z')G_2 \times [(x-x')\bar{M}_x + (y-y')\bar{M}_y + (z-z')\bar{M}_z]\} e^{-j\beta R} dx'dy'dz'$$

Far Field.

At far-zone the $\bar{E}$ & $\bar{H}$ field components are orthogonal to each other and form TEM (to r) mode fields.

If observation is taken in the far field ($\beta r >> 1$), the radial distance $\bar{R}$ from any point on the source or scatterer to the observation point can be assumed to be $\parallel^{el}$ to the radial distance 'r' from the origin to the observation point

$$R = [r^2 + (r')^2 + 2rr'\cos\psi]^{1/2}$$

Which can be approximated as:

$$R = \begin{cases} r - r'\cos\psi & \text{for phase variation} \\ r & \text{for amplitude variation} \end{cases}$$

ψ is angle between r and r'. These approximation provide a phase error of $\frac{\pi}{8}$ (22.5°) provided the observation are made at distances

$$r \geq \frac{2D^2}{\lambda}$$

D → largest dimension of radiator or scatterer.

This is the minimum distance to the far field.

Putting these assumptions into eqns for $\bar{A}$ & $\bar{F}$ we get

$$\bar{A} = \frac{\mu}{4\pi} \iint_s \bar{J}_s \frac{e^{-j\beta R}}{R} ds' \simeq \frac{\mu e^{-j\beta r}}{4\pi r} N$$

$$\bar{F} = \frac{\varepsilon}{4\pi} \iint_s \bar{M}_s \frac{e^{-j\beta R}}{R} ds' \simeq \frac{\varepsilon e^{-j\beta r}}{4\pi r} L$$

where

$$\bar{N} = \iint_S \bar{J}_s \, e^{j\beta r' \cos\psi} ds'$$

$$\bar{L} = \iint_S \bar{M}_s \, e^{j\beta r' \cos\psi} ds'$$

There is only θ & ϕ variation in the far field

$$\bar{E}_A = -j\omega\left[\bar{A} + \frac{1}{\beta^2}\nabla(\nabla\cdot\bar{A})\right]$$

$$\bar{H}_F = -j\omega\left[\bar{F} + \frac{1}{\beta^2}\nabla(\nabla\cdot\bar{F})\right]$$

$\bar{A}$ & $\bar{F}$ are defined above.

Since the second term generates variation of $\frac{1}{r^2}, \frac{1}{r^3}$ ---- and in the far field only $\frac{1}{r}$ is dominant. We can approximate the above equation as.

$$\bar{E}_A = -j\omega\bar{A}$$

$$\bar{H}_F = -j\omega\bar{F}$$

which can be written as

$$(E_A)_\theta = -j\omega A_\theta \qquad (E_A)_\phi = -j\omega A_\phi$$

$$(H_F)_\theta = \simeq -j\omega F_\theta \qquad (H_F)_\phi = -j\omega F_\phi$$

To find the remaining components we can use the relation

$$\bar{E}_F = -\frac{1}{\varepsilon}\nabla\times\bar{F} \qquad \bar{H}_A = \frac{1}{\mu}\nabla\times\bar{A}$$

We can also use the relation.

$$(E_F)_\theta = \eta (H_F)_\phi = -j\omega\eta F_\phi$$

$$(E_F)_\phi = -\eta (H_F)_\theta = +j\omega\eta F_\theta$$

$$(H_A)_\theta = \frac{-(E_A)_\phi}{\eta} = +j\omega \frac{A_\phi}{\eta}$$