ECE 5324/6324

NOTES

ANTENNA THEORY AND DESIGN

2013

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Text: Warren L. Stutzman and Gary A. Thiele, *Antenna Theory and Design*, Third Edition (2013), John Wiley & Sons.

The identified page numbers and the equations with dashes (x-xxx) refer to the equations of the text.

Example: Show that for far-field region $\nabla \times \vec{V} = -j\vec{\beta} \times \vec{V} = -j\beta\hat{r} \times \vec{V}$ for any vector $\vec{V}$ such as $\vec{A}, \vec{H}$ and $\vec{E}$ of the radiated fields from antennas.

Solution: The radiated fields $\vec{A}, \vec{H}$ and $\vec{E}$ are of the form

$$\vec{V} = \frac{K(\theta,\phi)}{r} e^{-jkr}\ \hat{V} \Rightarrow V(r,\theta,\phi)\hat{V} \tag{1}$$

where K(θ, ϕ) would, in general, depend upon the current distribution on the antenna.

In spherical coordinates

$$\begin{aligned}\nabla \times \vec{V} &= \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \times \Bigg[\overbrace{\frac{K(\theta,\phi)}{r} e^{-j\beta r}}^{V(r,\theta,\phi)}\ \hat{V}\Bigg] \\ &= \left(-\frac{1}{r} - j\beta\right) V(\hat{r} \times \hat{V}) + \frac{1}{r}\frac{\partial V}{\partial \theta}(\hat{\theta} \times \hat{V}) + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}(\hat{\phi} \times \hat{V})\end{aligned} \tag{2}$$

$$\cong -j\beta\, V(\hat{r} \times \hat{V}) = -j\beta\ \hat{r} \times \vec{V} = -j\vec{\beta} \times \vec{V} \tag{3}$$

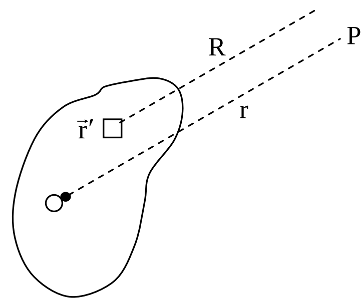
since all terms other than the second term in Eq. 2 are a factor of 1/βr smaller. For $\beta r = \frac{2\pi r}{\lambda} >> 1$, all of these terms can, therefore, be neglected.

This is a powerful relationship which can be applied for radiated fields from <u>any antenna.</u>

$$\vec{H} = \frac{\nabla \times \vec{A}}{\mu} = -\frac{j\vec{\beta} \times \vec{A}}{\mu} - j\frac{\beta}{\mu}\hat{r} \times \vec{A} \tag{4}$$

$$\vec{E} = \frac{\nabla \times \vec{H}}{j\omega\varepsilon} = -\frac{j\vec{\beta} \times \vec{H}}{j\omega\varepsilon} = -\frac{\beta}{\omega\varepsilon}\hat{r} \times \vec{H} = -\sqrt{\frac{\mu}{\varepsilon}}\ \hat{r} \times \vec{H} \tag{5}$$

p. 44, 45, 50 Text **General Theory of Conduction Current Antennas**



Formulate $\vec{J}(\vec{r}')$.

Steps

1. Calculate $\vec{A}$

$$\vec{A} = \underbrace{\frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}')}{R}e^{j(\omega t-\beta R)}dV'}_{\text{For volume current radiators}} = \underbrace{\frac{\mu}{4\pi}\int_{S'}\frac{\vec{J}_S}{R}e^{j(\omega t-\beta R)}dS'}_{\text{For surface current radiators}}$$

$$= \underbrace{\frac{\mu}{4\pi}\int_{\ell}\frac{\vec{I}}{R}e^{j(\omega t-\beta R)}d\ell}_{\text{For line current radiators}} \qquad (2\text{-}101)$$

2. $$\vec{H} = \frac{\nabla\times\vec{A}}{\mu_o} = -j\frac{\vec{\beta}\times\vec{A}}{\mu_o} = -j\frac{\beta\,\hat{r}\times\vec{A}}{\mu_o} = \frac{\nabla\times\vec{E}}{-j\omega\mu_o} = \frac{1}{\eta}\hat{r}\times\vec{E} \qquad (2\text{-}107)$$

3. $$\vec{E} = \frac{\nabla\times\vec{H}}{j\omega\varepsilon_o} = \frac{\beta^2\,\hat{r}}{j\omega\varepsilon_o\mu_o}\times(\hat{r}\times\vec{A}) = j\omega\,\hat{r}\times(\hat{r}\times\vec{A})$$

$$= -j\omega\left[\vec{A}-\left(\vec{A}\cdot\hat{r}\right)\hat{r}\right] = -j\omega\left(A_\theta\hat{\theta}+A_\phi\hat{\phi}\right) \qquad (2\text{-}105)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}$$

$$\hat{r}\times\left(\hat{r}\times\vec{A}\right) = \left(\hat{r}\cdot\vec{A}\right)\hat{r}-\left(\hat{r}\cdot\hat{r}\right)\vec{A}$$

From Eq. (2-105) we can write

$$\vec{E} = -j\omega\,\vec{A} \qquad (2\text{-}104)$$

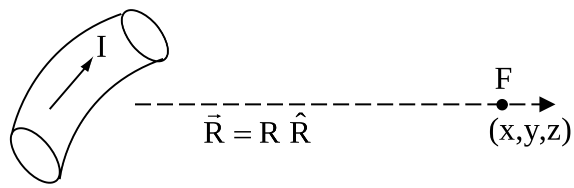
which is transverse to direction of propagation $\hat{r}$.

4. Calculate $$\vec{S} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right) \qquad (2\text{-}127)$$

5. Total Radiated Power = $$\int\vec{S}\cdot d\vec{s} = \frac{1}{2}\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}\left(E_\theta H_\phi^* - E_\phi H_\theta^*\right)r^2\sin\theta\,d\theta d\phi \qquad (2\text{-}128)$$

<u>Calculation of Magnetic Fields of Conduction Current Antennas</u>

Definition of Magnetic Vector Potential $\vec{A}$: A Simplifying Mathematical Intermediate Step



From Biot-Savart's law of electromagetism

$$\vec{B} = \mu_o \int_\ell \frac{I\, d\vec{\ell} \times \hat{R}}{4\pi R^2} = \mu_o \int_V \frac{\vec{J}\, dV' \times \hat{R}}{4\pi R^2} \quad \leftarrow -\nabla\left(\frac{1}{R}\right) \tag{1}$$

$$\vec{B} = \mu_o \, \nabla \times \int_{V'} \frac{\vec{J}\, dV'}{4\pi R} \tag{2}$$

In going from Eq. 1 to Eq. 2, we have used the following steps

$$\nabla\left(\frac{1}{R}\right) = -\frac{\hat{R}}{R^2} \tag{3}$$

$$\vec{J} \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R} \overset{0}{\cancel{\nabla \times \vec{J}(\vec{r}')}} - \nabla \times \left(\frac{\vec{J}}{R}\right) \tag{4}$$

The first term in Eq. 3 is zero, since the current density $\vec{J}$ is a function of source coordinates $\vec{r}' = (x', y', z')$ whereas the curl $\nabla \times \vec{J}$ involves derivatives with respect to field coordinates (x,y,z).

From Eq. 2

$$\vec{B} = \mu_o \, \nabla \times \int_{V'} \frac{\vec{J}\, dV'}{4\pi R} \equiv \nabla \times \vec{A} \tag{5}$$

Thus the magnetic field at the field point can be written as curl of magnetic vector potential $\vec{A}$ where $\vec{A}$ is given by

$$\vec{A} = \frac{\mu_o}{4\pi} \int_{V'} \frac{\vec{J}\, dV'}{R} \tag{6}$$

Note that calculation of $\vec{B}$ is a lot simpler if the intermediate step of first calculating $\vec{A}$ is undertaken since integral of Eq. 6 is much simpler than that of Eq. 5 or Eq. 1.

Note that because of time retardation for propagating fields, Eq. 6 should be modified to

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}')e^{j(\omega t-\beta R)}}{R}\, dV' \tag{7}$$

Same as Eq. 2-101
of the text

Once, the only complicated step that of integration for Eq. 7 has been done, the magnetic field $\vec{B}$ from Eq. 5 can be simplified to

$$\vec{B} = \nabla\times\vec{A} = -j\vec{\beta}\times\vec{A} = -j\beta\,\hat{R}\times\vec{A}$$

$$\vec{H} = -\frac{j\vec{\beta}\times\vec{A}}{\mu_o} \qquad \text{(2-107) text}$$

p. 45 Text **<u>A Uniform Line Source</u>**

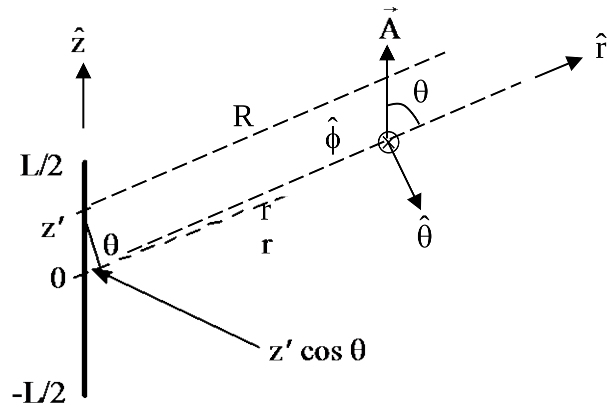


Fig. 2.9 text. A uniform line source.

$$I(z') = I_o \quad \text{for} \quad -\frac{L}{2} < z' < \frac{L}{2} \tag{2-109}$$

$$R = r - z' \cos\theta \tag{2-86}$$

$$\vec{A} = \frac{\mu}{4\pi r} e^{-j\beta r} \int_{-L/2}^{L/2} I_o \, e^{j\beta z' \cos\theta} dz' \quad \hat{z}$$

$$A_z = \frac{\mu I_o L e^{-j\beta r}}{4\pi r} \frac{\sin\left[\left(\frac{\beta L}{2}\right)\cos\theta\right]}{\left(\frac{\beta L}{2}\cos\theta\right)} \tag{2-110}$$

$$\vec{E} = j\omega \sin\theta \, A_z \, \hat{\theta} = j\omega \frac{\mu I_o L}{4\pi r} \sin\theta \frac{\sin\left[\left(\frac{\beta L}{2}\right)\cos\theta\right]}{\left(\frac{\beta L}{2}\cos\theta\right)} \quad \hat{\theta} \tag{2-111}$$

From Eq. (2-107)

$$\vec{H} = \frac{1}{\eta} E_\theta \, \hat{r} \times \hat{\theta} = \frac{E_\theta}{\eta} \hat{\phi}$$

Radiation pattern for a plot of normalized values of E(θ, φ) is given by

$$F(\theta, \phi) = \frac{|E(\theta, \phi)|}{|E_{max}|} \tag{2-112}$$

For an elemental or Hertzian dipole (βL/2 <<1)

$$F(\theta) = \sin\theta \tag{2-113}$$

Otherwise

$$F(\theta) = \sin\theta \frac{\sin\left[\left(\frac{\beta L}{2}\right)\cos\theta\right]}{\left(\frac{\beta L}{2}\cos\theta\right)} \tag{2-114}$$

Normalized Pattern factor $\quad P(\theta) = F^2(\theta) \qquad$ (2-119)

p. 34 **<u>An Infinitesimal (Hertzian) Current Dipole or An Ideal Dipole</u>**

$$I = I_o \qquad L = \Delta z$$

$$\frac{\beta L}{2} = \frac{\pi \Delta z}{\lambda} << 1$$

$$\frac{\beta L}{2}\cos\theta << 1$$

In Equations 2-110 to 2-114; $\frac{\sin x}{x} \cong 1$

$$\vec{E} = j\omega \frac{\mu I_o \Delta z}{4\pi r}\sin\theta\,\hat{\theta} \tag{2-74a}$$

$$\vec{H} = \frac{E_\theta}{\eta}\hat{\phi} = \frac{j\beta I_o \Delta z}{4\pi r}\sin\theta \quad \hat{\phi} \tag{2-74b}$$

$$\vec{S} = \frac{I_o^2}{8}\eta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{\sin^2\theta}{r_o^2} \quad \hat{r} \tag{2-76}$$

$$\text{Radiated Power } P = \frac{1}{2}\iint \mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)\cdot d\vec{s}$$

$$= \frac{1}{2}\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}\left(E_\theta H_\phi^* - E_\phi H_\theta^*\right) r^2 \sin\theta\, d\theta\, d\phi \tag{2-128}$$

$$= \frac{I_o^2}{3}\pi\eta\left(\frac{\Delta z}{\lambda}\right)^2 = \frac{1}{2} I_o^2 R_r$$

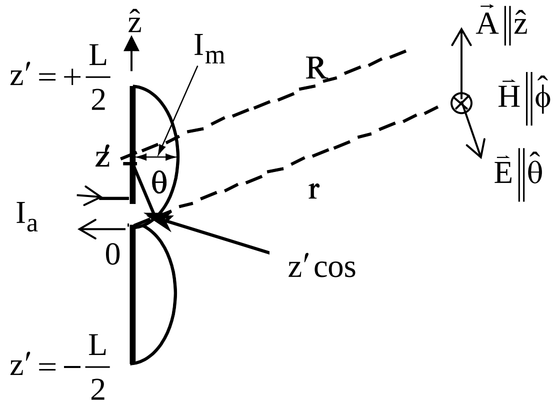
Valid only for very short length Hertzian dipoles $(\Delta z \le 0.02\lambda)$

789.6

$$R_r = R_a = \frac{2}{3}\pi\eta\left(\frac{\Delta z}{\lambda}\right)^2 = 80\,\pi^2\left(\frac{\Delta z}{\lambda}\right)^2\ \Omega \qquad \text{(2-169)} \quad \text{p. 57}$$

$$D = \frac{S_{max}}{S_o} = 1.5 \qquad \text{(2-148)} \quad \text{p. 54}$$

<u>A Linear Center-Fed Dipole</u>
(See also pp. 152-160 of the text)



$$I(z') = I_m \sin\left[\beta\left(\frac{L}{2} - |z'|\right)\right] \qquad |z'| < \frac{L}{2} \qquad \text{(6-1)} \quad \text{p. 152}$$

Note that $$I_a = I(z')\big|_{z'=0} = I_m \sin\left(\beta\frac{L}{2}\right) \qquad (1)$$

As previously assumed on p. 3 of these Notes (from Eq. 2-86 of the text),

$$R = r - z'\cos\theta$$

From Eq. 2-101

$$\vec{A} = \frac{\mu}{4\pi r} I_m \left[\int_{-L/2}^{0} \sin\left[\beta\left(\frac{L}{2} + z'\right)\right] e^{j\beta z'\cos\theta} dz' + \int_{0}^{L/2} \sin\beta\left(\frac{L}{2} - z'\right) e^{j\beta z'\cos\theta} dz'\right] e^{-j\beta r}\,\hat{z} \qquad \text{(6-3)}$$

$$\vec{E} = j\omega\sin\theta\, A_z\,\hat{\theta} = j\eta\frac{e^{-j\beta r}}{4\pi r} 2I_m\, F(\theta)\,\vec{\theta} = \boxed{\frac{j60\, I_m}{r} F(\theta) e^{-j\beta r}\,\hat{\theta}} \qquad \text{(6-6)} \quad \text{p. 154}$$

where $F(\theta)$ is the function that gives the variation of radiated fields with angle θ. Note that this expression for the radiated θ-directed E field can also be expressed in terms of the feedpoint antenna current I_a using Eq. (1) on this page.

$$F(\theta) = \frac{\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)}{\sin\theta} \tag{2}$$

See p. 154 of the text, Fig. 6-4, for plots of F(θ) for several values of L/λ.

F(θ) is always zero for angle $\theta = 0^\circ$ i.e. no radiated fields along the length of the dipole.

$$\vec{S} = \frac{1}{2}\text{Re}\left(E_\theta H_\phi^*\right) = \frac{E_\theta E_\theta^*}{2\eta}\hat{r} = \frac{15\, I_m^2}{\pi r^2} F^2(\theta)\,\hat{r}$$

$$= \frac{15}{\pi r^2}\,\frac{I_a^2}{\sin^2\left(\beta\frac{L}{2}\right)} F^2(\theta)\hat{r} = \frac{30}{\pi r^2}\,\frac{P_{rad}}{R_a}\cdot\frac{F^2(\theta)}{\sin^2\left(\beta\frac{L}{2}\right)} \tag{3}$$

$$\text{Radiated Power } P_{rad} = \iint \vec{S}\cdot d\vec{s} = \frac{15\, I_m^2}{\pi r^2}\int_0^{2\pi}\int_0^{\pi} F^2(\theta) r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2} I_m^2\, R_{rm} = \frac{1}{2} I_a^2\, R_a \tag{4}$$

$$R_a = \frac{60\, I_m^2}{I_a^2}\int_0^{\pi} F^2(\theta)\sin\theta\, d\theta \tag{5}$$

Where R_a is the antenna equivalent resistance at the feed point ($z' = 0$).

The antenna equivalent resistance R_a is given by Table l on p. 7.
Thus the directivity D of a linear center-fed antenna of end-to-end length L is given by:

$$D = \frac{S_{max}}{S_o} = \frac{120}{R_a}\,\frac{F^2(\theta)\big|_{max}}{\sin^2\left(\frac{\beta L}{2}\right)} \tag{6}$$

Table 1. Calculated values of the driving point resistance R_a for end-fed monopoles of different lengths h/λ. **(Multiply by 2 to obtain the driving point resistance for center-fed dipoles of length L = 2h.)**

h/λ = L/2λ	R_a	h/λ = L/2λ	R_a
1.00E-02	3.9499E-02	5.10E-01	2.4554E 04
2.00E-02	1.5824E-01	5.20E-01	5.9649E 03
3.00E-02	3.5699E-01	5.30E-01	2.5708E 03
4.00E-02	6.3701E-01	5.40E-01	1.3996E 03
5.00E-02	1.0001E 00	5.50E-01	8.6554E 02
6.00E-02	1.4486E 00	5.60E-01	5.7994E 02
7.00E-02	1.9856E 00	5.70E-01	4.1067E 02
8.00E-02	2.6146E 00	5.80E-01	3.0288E 02
9.00E-02	3.3400E 00	5.90E-01	2.3056E 02
1.00E-01	4.1669E 00	6.00E-01	1.8009E 02
1.10E-01	5.1013E 00	6.10E-01	1.4382E 02
1.20E-01	6.1503E 00	6.20E-01	1.1718E 02
1.30E-01	7.3219E 00	6.30E-01	9.7309E 01
1.40E-01	8.6256E 00	6.40E-01	8.2337E 01
1.50E-01	1.0072E 01	6.50E-01	7.1026E 01
1.60E-01	1.1674E 01	6.60E-01	6.2519E 01
1.70E-01	1.3447E 01	6.70E-01	5.6217E 01
1.80E-01	1.5407E 01	6.80E-01	5.1692E 01
1.90E-01	1.7574E 01	6.90E-01	4.8638E 01
2.00E-01	1.9971E 01	7.00E-01	4.6837E 01
2.10E-01	2.2626E 01	7.10E-01	4.6134E 01
2.20E-01	2.5571E 01	7.20E-01	4.6422E 01
2.30E-01	2.8844E 01	7.30E-01	4.7637E 01
2.40E-01	3.2490E 01	7.40E-01	4.9746E 01
2.50E-01	3.6564E 01	7.50E-01	5.2747E 01
2.60E-01	4.1131E 01	7.60E-01	5.6664E 01
2.70E-01	4.6272E 01	7.70E-01	6.1554E 01
2.80E-01	5.2383E 01	7.80E-01	6.7501E 01
2.90E-01	5.8687E 01	7.90E-01	7.4630E 01
3.00E-01	6.6233E 01	8.00E-01	8.3108E 01
3.10E-01	7.4914E 01	8.10E-01	9.3155E 01
3.20E-01	8.4974E 01	8.20E-01	1.0506E 02
3.30E-01	9.6727E 01	8.30E-01	1.1923E 02
3.40E-01	1.1059E 02	8.40E-01	1.3617E 02
3.50E-01	1.2711E 02	8.50E-01	1.5658E 02
3.60E-01	1.4706E 02	8.60E-01	1.8143E 02
3.70E-01	1.7148E 02	8.70E-01	2.1205E 02
3.80E-01	2.0186E 02	8.80E-01	2.5036E 02
3.90E-01	2.4040E 02	8.90E-01	2.9918E 02
4.00E-01	2.9042E 02	9.00E-01	3.6280E 02
4.10E-01	3.5715E 02	9.10E-01	4.4794E 02
4.20E-01	4.4924E 02	9.20E-01	5.6578E 02
4.30E-01	5.8183E 02	9.30E-01	7.3589E 02
4.40E-01	7.8350E 02	9.40E-01	9.9525E 02
4.50E-01	1.1136E 03	9.50E-01	1.4208E 03
4.60E-01	1.7137E 03	9.60E-01	2.1960E 03
4.70E-01	2.9936E 03	9.70E-01	3.8530E 03
4.80E-01	6.6033E 03	9.80E-01	8.5364E 03
4.90E-01	2.5836E 04	9.90E-01	3.3546E 04
5.00E-01	3.2730E 33	1.00E 00	1.0671E 33

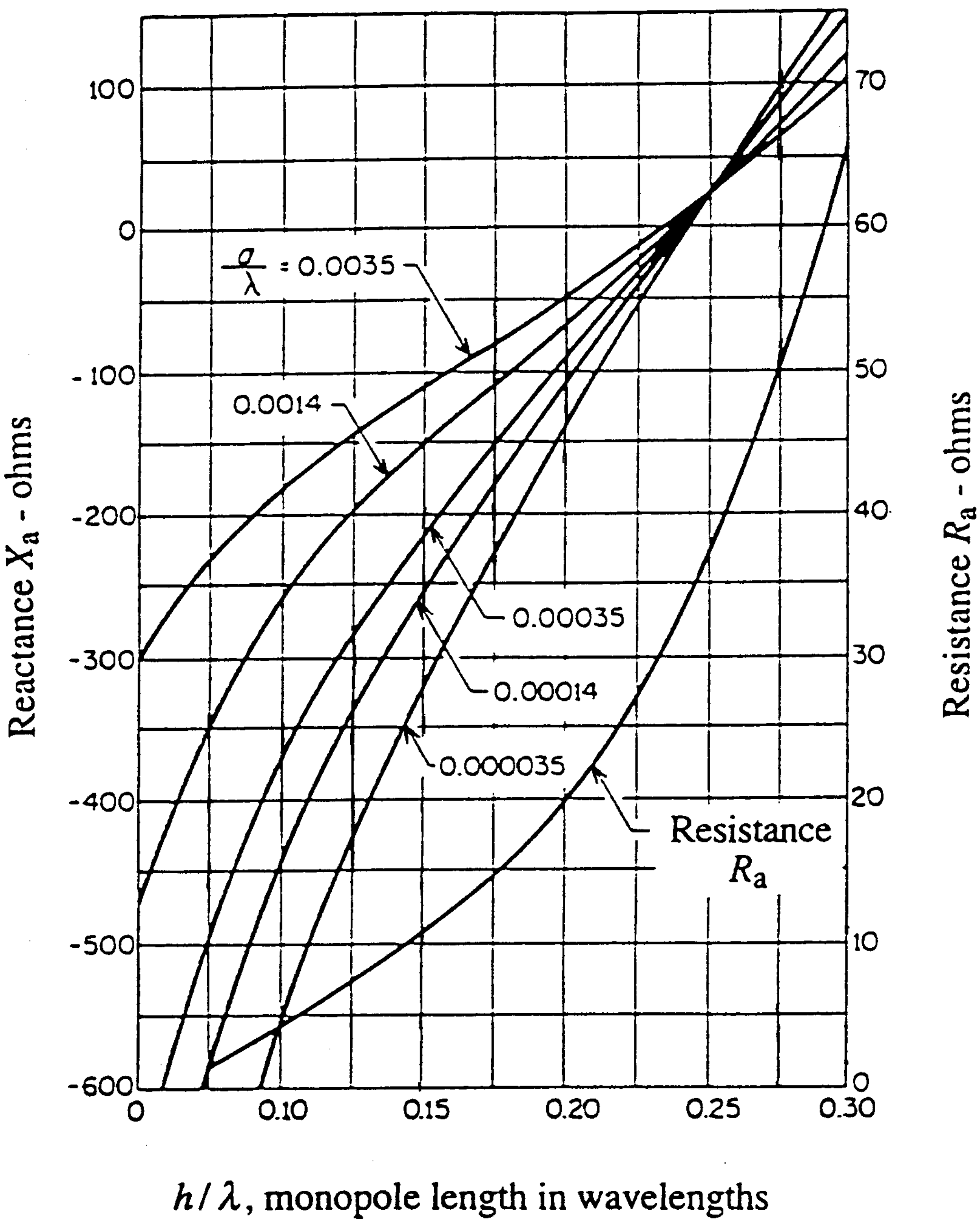


Fig. 1. The calculated resistance R_a and reactance X_a of an **end-fed monopole** antenna of length h (in terms of wavelength λ). Multiply by 2 **to obtain the driving point resistance R_a for a center-fed dipole** antenna of length L = 2h.

Example 1: Calculate and compare the directivities, gains, and power densities including E-fields created by dipole antennas of lengths L = 0.07 λ, 0.18 λ, 0.5 λ, and 1.1 λ. Power radiated by the antenna is 100 W and distance from the antenna to the field point r_o = 10 km.

Note that the radiated power and the distance r_o are needed to calculate the power density and maximum electric fields.

							Including Ohmic losses for Prob. 5 of HW		
L/2 λ	From p. 7 of Notes R_a ohms	I_a^* A	$F(\theta)\big\|_{\theta=90^\circ}$ [i]	D[ii]	$S_{\max}\big\|_{\theta=90^\circ} = S_o D = \left(\frac{P_{rad}}{4\pi r^2}\right) D$ μW/m²	$S = \frac{E^2}{2\eta}$, $E_{\max} = \sqrt{2\eta S_{\max}}$ mV/m	R_{ohmic}[iii] Ω	From Eq. 2-153 antenna efficiency $e_r = \frac{R_a}{R_a + R_{ohmic}}$	From Eq. 2-155 $G = e_r D$
0.035	0.994	14.18	0.0241	1.47	0.117	9.39	0.0388	0.9624	1.414
0.09	6.68	5.47	1.844	1.516	0.1206	9.54	0.1042	0.9846	1.493
0.25	73.12	1.65	1.0	1.64	0.1305	9.92	0.208	0.997	1.635
0.55	1731.1	0.34	1.951	2.76	0.2196	12.87	8.758	0.995	2.746

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* Note that $P_{rad} = \left(\frac{1}{2}\right) I_a^2 R_a$ from Eq. (4) on p. 6 of Class Notes; $I_a = \sqrt{\frac{2P_{rad}}{R_a}}$

[i] From Eq. (2) on p. 6 of Class Notes, $F(\theta) = \dfrac{\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)}{\sin\theta} = \dfrac{\cos\left(\frac{\pi L}{\lambda}\cos\theta\right) - \cos\left(\frac{\pi L}{\lambda}\right)}{\sin\theta}$. For L/λ < 1.38 – 1.4; F(θ) is max. for θ = 90°.

$$F(\theta)\Big|_{\substack{\max \\ \theta=90^\circ}} = 1 - \cos(\pi L/\lambda)$$

[ii] From Eq. (6) on p. 6 of Notes, $D = \dfrac{120}{R_a} \dfrac{F^2(\theta)\big|_{\max}}{\sin^2(\pi L/\lambda)}$

[iii] R_{ohmic} is given by Eq. (9) on p. 13 of Class Notes; $R_S = 1.988\sqrt{f_{MHz}/\sigma}$; Take 2a = 3.264 mm (0.1285) ← 8 AWG wire (App. B on p. 783 of the Text).

For Aluminum, from App. B.1, σ = 3.5 × 10^7 S/m; take f_{MHz} = 10 MHz.

Example 2:

P_{rad} = 1 W; f = 835 MHz; r = 1 km

$$L = 2h = 0.65\lambda = 0.65\frac{30}{0.835} = 23.35\text{ cm}$$

$$\frac{h}{\lambda} = \frac{L}{2\lambda} = 0.325$$

$$R_a\Big|_{\text{Table1}} = 2\times\frac{84.974+96.727}{2} = 181.7\Omega$$

$$P_{rad} = \frac{1}{2}I_a^2\ R_a$$

$$I_a = 0.1049\text{ A}$$

$$I_m = \frac{I_a}{\sin\left(\frac{\beta L}{2}\right)} = \frac{I_a}{\sin\left(\frac{\pi L}{\lambda}\right)} = 0.1177$$

$$|E_{max}| = \frac{60\ I_m}{r}\ F(\theta)\Big|_{\theta=90^\circ} = \frac{60\ I_m}{r}\left\{\frac{1-\cos\left(\frac{\beta L}{2}\right)}{1}\right\} = 10.265\quad \frac{\text{mV}}{\text{m}}$$

$$E_{rms} = 0.707\ E_{peak} = 7.26\quad \frac{\text{mV}}{\text{m}}$$

$$S_{max} = \frac{|E_{max}|^2}{2\eta} = \frac{|E_{rms}|^2}{\eta} = 0.1398\ \frac{\mu\text{W}}{\text{m}^2}$$

$$D = \frac{120}{R_a}\ \frac{F^2(\theta)\Big|_{max}}{\sin^2\left(\frac{\beta L}{2}\right)} = \frac{120}{181.7}\times\frac{2.114}{\sin^2(0.65\pi)} = 1.759\ (2.45\text{ dBi})$$

This is an improvement of only 1.073 times (or 0.3 dB) relative to a half wave dipole.

$$S_{max} = S_o\ D = 0.1398\ \frac{\mu\text{W}}{\text{m}^2}$$

Radiation patterns xy plane H-plane

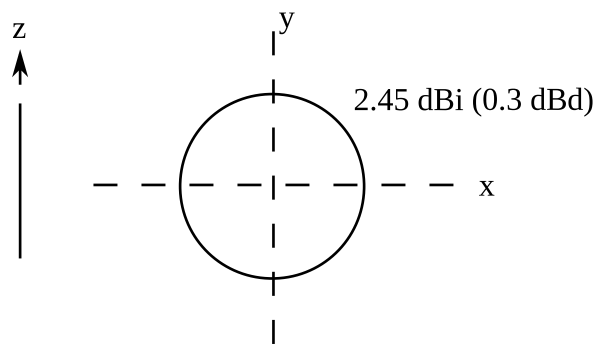


Fig. 2. The radiation pattern of a z-directed dipole antenna for the xy plane or H-plane (normal to the orientation of the dipole).

See also p. 154 (Fig. 6-4)

yz plane (E-plane)

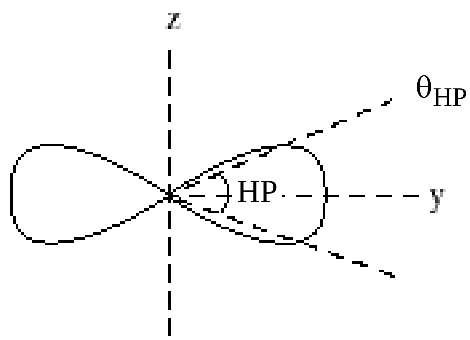


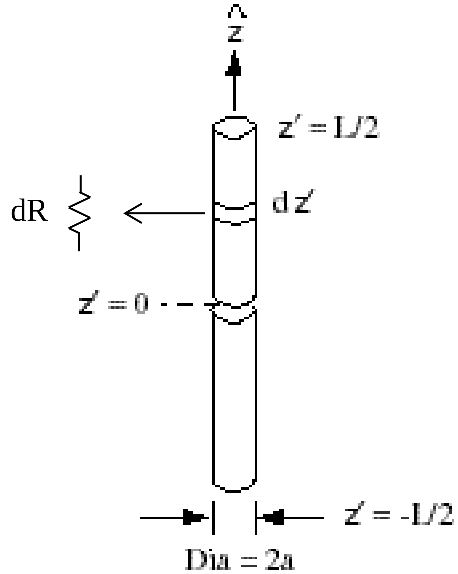
Fig. 3. The radiation pattern of the z-directed dipole antenna for the yz plane or the E-plane.

$$\boxed{HP = 2 \times (90° - \theta_{HP})} \qquad (2\text{-}126)$$

p. 49

$$\left[\frac{\cos\left(\frac{\beta L}{2}\cos\theta_{HP}\right)-\cos\left(\frac{\beta L}{2}\right)}{\sin\theta_{HP}}\right]^2 = \frac{1}{2}\left[1-\cos\left(\frac{\beta L}{2}\right)\right]^2 \qquad (7)$$

pp. 57, 58 Text **Ohmic Losses for a Linear Dipole**



From Eq. 6-1, p. 152 Text

$$I(z') = I_m \sin\left[\beta\left(\frac{L}{2} - z'\right)\right] \qquad 0 \le z' \le \frac{L}{2}$$
$$= I_m \sin\left[\beta\left(\frac{L}{2} + z'\right)\right] \qquad -\frac{L}{2} \le z' \le 0 \tag{1}$$

$$I_a = I(z')\big|_{z'=0} = I_m \sin\left(\frac{\beta L}{2}\right) \tag{2}$$

Ohmic power lost in the antenna

$$P_{ohmic} = \frac{1}{2}\int_{-L/2}^{L/2}\left(I^2 dR\right) \tag{3}$$

$$dR = \frac{dz'}{(2\pi a\,\delta)\sigma} = \frac{R_s\,dz'}{2\pi a} \tag{4}$$

where

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{5}$$

(2-171)
p. 58

is the surface resistance which depends on the conductivity σ of the material and frequency ω (= $2\pi f$). See App. B.1 of the Text for σ of various metals.

$$P_{ohmic} = \frac{R_s}{4\pi a}\left[\int_0^{L/2} I_m^2 \sin^2 \beta\left(\frac{L}{2} - z'\right)dz' + \int_{-L/2}^{0} I_m^2 \sin^2 \beta\left(\frac{L}{2} + z'\right)dz'\right] \tag{6}$$

$$\int_0^{L/2} \sin^2 \beta\left(\frac{L}{2} + z'\right) dz' = \frac{1}{\beta}\int_0^{\beta L/2} \sin^2 \zeta \, d\zeta = \frac{1}{2\beta}\left[\zeta - \frac{\sin(2\zeta)}{2}\right]\Bigg|_0^{\beta L/2}$$

$$= \frac{1}{2\beta}\left[\frac{\beta L}{2} - \frac{\sin(\beta L)}{2}\right] \tag{7}$$

where $\zeta = \ = \beta\left(\frac{L}{2} + z'\right)$

$$P_{ohmic} = I_m^2 \frac{R_s L}{8\pi a}\left[1 - \frac{\sin(\beta L)}{(\beta L)}\right] = \frac{1}{2} I_A^2 R_{ohmic} \tag{8}$$

$$R_{ohmic} = \frac{R_s L}{4\pi a} \frac{1}{\sin^2\left(\frac{\beta L}{2}\right)}\left[1 - \frac{\sin(\beta L)}{(\beta L)}\right] \tag{9}$$

Antenna efficiency $e_r = \frac{P}{P_{in}} = \frac{P_{rad}}{P_{rad} + P_{ohmic}} = \frac{R_a}{R_a + R_{ohmic}}$ (2-177)

Gain G = $e_r D$ (2-155)

<u>For a Short Dipole ($L = \Delta z << \lambda$)</u>

$$R_a = 20\,\pi^2 \left(\frac{\Delta z}{\lambda}\right)^2 \tag{2-172}$$

$$R_{ohmic} = \frac{R_s \Delta z}{6\pi} = \frac{R_s L}{6\pi a} \tag{2-175}$$

For the general case of a linear dipole or a monopole R_{ohmic} is calculated from the general Eq. 9 given above.

Example 3:

<u>For a Short Dipole</u>

$$R_a = 20\,\pi^2 \left(\frac{L}{\lambda}\right)^2 \cong 197.4\left(\frac{L}{\lambda}\right)^2 \tag{2-172}$$

See e.g. Table 1 on page 7 for $L/\lambda = 0.02$, $R_a = 2 \times 0.0394 = 0.0788\ \Omega$.

Using the conductivity of steel (see App. B.1 of the Text) $\sigma = 2 \times 10^6$ S/m.

From Eq. 2-171 or Eq. 5

$$R_s = 1.4 \times 10^{-3} \sqrt{f_{MHz}} \quad \Omega$$

From Eq. 9, βL is small and we can expand sin x for small x

$$R_{ohmic} = \frac{R_s L}{4\pi a} \frac{1}{\left(\frac{\beta L}{2}\right)^2} \left[1 - \frac{\beta L - \frac{(\beta L)^3}{6}}{\beta L} \right] = \frac{R_s L}{6\pi a} \qquad \text{Short dipole} \qquad (2\text{-}175)$$

p. 59

For $L/\lambda = 0.02$ dipole at f = 1 MHz; taking 2a = l/8"

$$\lambda = 300\text{m} \quad ; \quad L = 6\text{m}$$

$$R_{ohmic} = \frac{1.4 \times 10^{-3} \times 6}{6\pi \times \frac{1}{16} \times 2.54 \times 10^{-2}} = 0.2807\Omega$$

Antenna Efficiency $\quad e_r = \frac{R_a}{R_a + R_{ohmic}} = \frac{0.0788}{0.0788 + 0.2807} = 0.219 \qquad (21.9\%)$

Gain G = $e_r D = 0.219 \times 1.5 = 0.3285$.

Note that for short dipoles of thin wires, the ohmic resistance can be substantial and even larger than R_a. Therefore, this leads to reduced efficiency of radiation.

Example 4:

<u>For a Half Wave Dipole</u>

$L = 0.5\lambda$; $R_a = 73.12\ \Omega$; $\frac{\beta L}{2} = \frac{\pi L}{\lambda} = \frac{\pi}{2}$; $\beta L = \pi$; f = 10 MHz; L = 15m; 2a = 1/8"

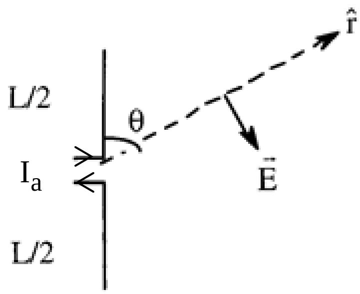
From Eq. 9

$$R_{ohmic} = \frac{R_s L}{4\pi a} = \frac{\left(1.4 \times 10^{-3} \sqrt{10}\right) \times 15}{4\pi \times \left(\frac{1}{16} \times 2.54 \times 10^{-2}\right)} = 3.33\Omega$$

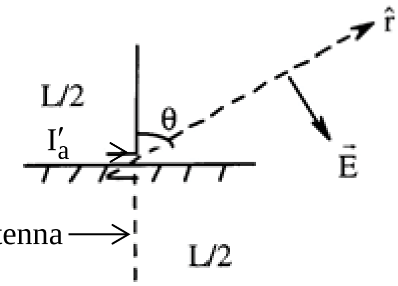
$$e_r = \frac{73.12}{73.12 + 3.33} = 0.9565 \qquad (95.65\%)$$

$$G = 0.9565\, D = 0.9565 \times 1.64 = 1.568$$

pp. 75-81 Text **Dipoles Versus Monopoles Above a Perfect Ground or Reflector**



a. A dipole.



b. The corresponding monopole above ground or reflector.

I(z′) same as on page 5 of the Notes $$I(z') = I'_m \sin\beta\left(\frac{L}{2} - z'\right) 0 \le z' \le \frac{L}{2} \quad (1)$$

For $0 \le \theta \le 180^\circ$ For $0 \le \theta \le 90^\circ$

$$\vec{E} = j\frac{60 I_m}{r} F(\theta) e^{-j\beta r}\hat{\theta} \quad (2) \qquad \vec{E}' = \frac{j60 I'_m}{r} F(\theta) e^{-j\beta r}\hat{\theta} \quad (3)$$

$$\vec{S} = \frac{15 I_m^2}{\pi r^2} F^2(\theta)\hat{r} \quad (4) \qquad \vec{S}' = \frac{15 I_m'^2}{\pi r^2} F^2(\theta)\hat{r} = \frac{15}{\pi r^2}\frac{I_a'^2}{\sin^2\left(\frac{\beta L}{2}\right)} F^2(\theta)\hat{r} \quad (5)$$

$$P_{rad} = \frac{1}{2} I_a^2 R_a \quad (6) \qquad P'_{rad} = \frac{1}{2} I_a'^2 R'_a \quad (7)$$

F(θ) is given as Eq. (2) on p. 6 of the Notes.

Since a monopole radiates in the upper half space while a dipole radiates both in the upper and lower half spaces,

$$S_{dipole} = \frac{1}{2} S'_{monopole} \text{ for } \textbf{identical radiated powers} \quad (8)$$

$$D'_{monopole} = 2 D_{dipole} \quad (9)$$

$$\left.\begin{matrix} R'_a \\ X'_a \end{matrix}\right|_{monopole} = \frac{1}{2} \left.\begin{matrix} R_a \\ X_a \end{matrix}\right|_{dipole} \quad (10)$$

For identical radiated powers

$$I'_a = \sqrt{2} I_a \quad (11)$$

$$I'_m = \sqrt{2} I_m \quad (12)$$

Example 4:

$$\frac{h}{\lambda} = \frac{L}{2\lambda} = 0.35 \quad \text{Monopole Antenna}$$

$$f = 1.5\ \text{MHz}, \quad \lambda = 200\ \text{m}\ ; \quad r = 1\ \text{km}$$

$$h = \frac{L}{2} = 70\ \text{m}\ ; \quad P_{rad} = 10^3\ \text{W}\ (1\ \text{KW})$$

From Table 1 on page 7, $R'_a = 127.1\,\Omega$ **(do not multiply by 2 for monopoles)**

From Eq. 7, $I'_a = 3.967\text{A}$

From Eq. 5,

$$S'\Big|_{\text{monopole}} = \frac{15}{\pi \times 10^6} \times \frac{(3.967)^2}{\sin^2(0.7\,\pi)} \left. \frac{\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)}{\sin\theta} \right|^2_{\theta = 90^\circ} = 0.29\ \text{mW}/\text{m}^2$$

$$D'\Big|_{\text{monopole}} = \frac{S'}{S_o = \frac{P_{rad}}{4\pi r^2}} = 3.64 = 2 \times D\Big|_{\text{dipole}}$$

From Eq. (6) on p. 6 of Class Notes

$$D = \frac{120}{R_a} \frac{F^2(\theta)\big|_{\max}}{\sin^2\left(\frac{\beta L}{2}\right)}$$

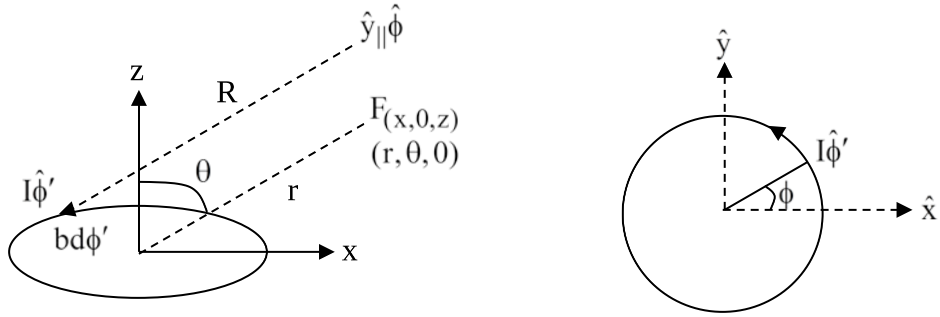
Note that $\frac{L}{2} = h$ which is the height of the monopole.

pp. 84-89 **<u>Small Diameter (<< λ) Loop Antennas</u>**

The loop antenna is a radiating (or receiving) coil of one or more turns of circular or rectangular form. Ferrite or air core loops are used extensively in radio receivers, direction finders, aircraft receivers, and UHF transmitters.

The theory of loop antennas is derived in a manner similar to the General Theory of Conduction Current Antennas given on page 44 of Text and on page 2 of my handout notes.

We start by assuming, as seen in Fig. 1, that the current I in the loop has the same magnitude and phase. This is certainly possible for small diameter loops where $2\pi b < \lambda/10$.



Loop in the xy plane

Fig. 1. A circular loop antenna of radius 'b'.

From General Theory of Conduction Current Antennas, from Eq. 2-101,

$$\vec{A} = \frac{\mu}{4\pi}\int_0^{2\pi} \frac{I\,\hat{\phi}' e^{-j\beta R}}{R}\, b\,d\phi' \tag{1}$$

$$R = \sqrt{(x - x')^2 + (y - y')^2 + z^2} \tag{2}$$

$$x' = b\cos\phi';\quad y' = b\sin\phi';\quad z' = 0;\quad x = r\sin\theta;\quad y = 0;\quad z = r\cos\theta \tag{3}$$

Note that we have defined the x-axis (the choice of which is arbitrary) such that the field point lies in the xz plane. The field point F, therefore, has coordinates (x, 0, z) in Cartesian coordinate system and (r, θ, 0) is spherical coordinate system.

Substituting Eq. 3 into Eq. 2

$$\begin{aligned} R &= \left[r^2 + b^2 - 2\,br\sin\theta\cos\phi'\right]^{1/2} \cong r\left[1 - \frac{b}{r}\sin\theta\cos\phi'\right] \\ &= r - b\sin\theta\cos\phi' \end{aligned} \tag{4}$$

since, for the far-field region, r >> b.

Using the far-field approximation for Eq. 1

$$\vec{A} = \frac{\mu I\, e^{-j\beta r}}{2\pi r} \int_0^{2\pi} \hat{\phi}'\, e^{j\beta b \sin\theta \cos\phi'} b\, d\phi' \tag{5}$$

For small radii $\beta b = \frac{2\pi b}{\lambda} << 1$, we can write

$$e^{j\beta b \sin\theta \cos\phi'} = 1 + j\beta b \sin \cos\phi' \tag{6}$$

We can also write (see Fig. 1(b))

$$\hat{\phi}' = -\hat{x}\sin\phi' + \hat{y}\cos\phi' \tag{7}$$

Also to be noted is that for the field point F

$$\hat{y} = \hat{\phi} \tag{8}$$

From Eq. 5 therefore, we can write

$$A = \frac{j\mu I\, S}{4\pi r} \beta e^{-j\beta r} \sin\theta\, \hat{\phi} \tag{9}$$

(p. 86 Text Eq. 3-49)

where $S = \pi b^2$ is the area of the loop.

On page 19, we compare the expressions for the radiated fields from a loop antenna to those for an ideal (infinitesimal) dipole and show duality of the two sets of fields. Ohmic resistance of a **circular** loop antenna can be written as follows:

$$R_{ohmic} = R_w = \frac{2\pi b}{(2\pi a \delta)\sigma} = \frac{bR_s}{a} \tag{10}$$

(3-60)
p. 88 Text

where

$$R_s = \frac{1}{\sigma\delta_s} = 1.988\sqrt{\frac{f_{MHz}}{\sigma}}$$

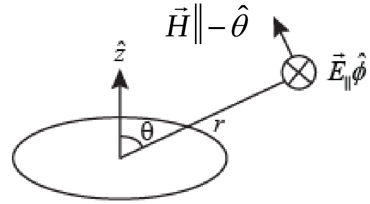
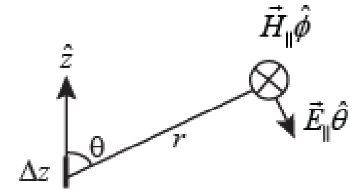
where "b" is the mean loop radius and "a" is the wire radius; $R_s = 1/\sigma\delta$ is the surface resistance at the frequency of interest previously defined on page 12 of Class Notes.

The small loop antenna is inherently inductive. For a small circular loop of N turns wound on a magnetic core

$$L = N^2 b\, \mu_{eff}\, \mu_o \left[\ell n\left(\frac{8b}{a}\right) - 2\right] \tag{11}$$

(Eq. 3-62 p. 88 Text)

Table 2. Field expressions for small diameter circular loop antennas and an ideal (infinitesimal) dipole antenna [see p. 4 of Class Notes].

	Loop Antenna	Ideal (Infinitesimal) Dipole
		
Magnetic Vector Potential $(\vec{A})$	$\frac{jIS}{4\pi r}\beta \sin\theta e^{-j\beta r}\hat{\phi}$ (9) (3-48) Text	$\frac{\mu I \Delta z}{4\pi r}e^{-j\beta r}\hat{z}$ (2-65) p. 33 Text
Magnetic Field $\vec{H} = \frac{\nabla\times\vec{A}}{\mu} = -\frac{j\beta}{\mu}\hat{r}\times\vec{A}$	$-\frac{IS}{4\pi r}\beta^2 e^{-j\beta r}\sin\theta\hat{\theta}$ (3-50) p. 86 Text	$\frac{j\beta I\Delta z}{4\pi r}e^{-j\beta r}\sin\theta\hat{\phi}$ (2-74b) since $\hat{r}\times\hat{z} = -\sin\theta\hat{\phi}$ (2-70) p. 33 Text
Electric Field $\vec{E} = \frac{\nabla\times\vec{H}}{j\omega\varepsilon} = -\eta\hat{r}\times\vec{H}$	$\frac{\eta IS}{4\pi r}\beta^2 e^{-j\beta r}\sin\theta\hat{\phi}$ (3-49) p. 86 Text	$\frac{j\eta I\Delta z}{4\pi r}\beta e^{-j\beta r}\sin\theta\hat{\theta}$ (2-74a) p. 34 Text
Radiated Power Density $\vec{S} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right) = \frac{\vec{E}\bullet\vec{E}^*}{2\eta}\hat{r}$	$\frac{\eta I^2 S^2}{32\pi^2 r^2}\beta^4\sin^2\theta\hat{r}$	$\frac{\eta I^2(\Delta z)^2}{2(4\pi r)^2}\beta^2\sin^2\theta\hat{r}$ (2-76)
Radiated Power $P = \int\vec{S}\bullet d\vec{s} \equiv \frac{1}{2}I^2 R_r$	$10I^2\left(\beta^2 S\right)^2$ (3-52) p. 86 Text	$\frac{\omega\mu\beta}{12\pi}(I\Delta z)^2$ (2-77) p. 33 Text
R_r (for single turn loop)	$20\left(\beta^2 S\right)^2 \cong 31,200\left(\frac{S}{\lambda^2}\right)^2\ \Omega$ (3-53)	$80\pi^2\left(\frac{\Delta z}{\lambda}\right)^2 = 790\left(\frac{\Delta z}{\lambda}\right)^2$ (2-169) p. 57 Text
Directivity $D = \frac{S_{\max}}{S_o}$	1.5	1.5
$R_r\big\vert_{N-\text{turn loop}}$	$31,200\left(\mu_{eff}\frac{NS}{\lambda^2}\right)^2\ \Omega$ (3-54)	

For an N-turn loop, R_{ohmic} is also higher proportional to overall length of the wire

$$R_{ohmic}\Big|_{\substack{N-turn\\ loop}} = N\frac{bR_s}{a} \tag{12}$$

The effective permeability μ_{eff} depends not only on the permeability μ_r of the ferrite core material, but also on the core geometry, i.e., length to diameter ratio R, given as follows:

$$\mu_{eff} = \frac{\mu_r}{1+D\left(\mu_r - 1\right)} \tag{13}$$

where [4]D is the demagnetization factor approximately given by D [4]

$$D \simeq 0.37R^{-1.44} \tag{14}$$

p. 87 Text

Example 5 (see also Ex. 3-1, p. 88, Text):

Calculate the input impedance, directivity, and gain for an N = 1000 turn loop antenna wound with a AWG 22 copper wire on a ferrite rod of diameter 3/4". This antenna is to be used at a frequency of 1.5 MHz. It is given that $\mu_{eff} = 50$ for the ferrite that is used.

Solution: From p. 783 of the Text, for AWG 22 wire d = 2a = 0.644 mm ⇒ 0.0253″

From p. 58, Eq. 2-171

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}} = 1.988\sqrt{\frac{f_{MHz}}{\sigma}} \text{ ohms} \tag{15}$$

For copper $\sigma = 5.7 \times 10^7$ S/m (p. 783, Text); $R_s = 3.22 \times 10^{-4}\ \Omega$ at f = 1.5 MHz

$$\text{Mean loop radius } b = \frac{3''}{8} + a = 9.847 \text{ mm}$$

From Eq. 12, for N = 1000-turn loop

$$R_{ohmic} = 9.85\Omega$$

From Eq. 3-54 Text (see also p. 19 of Class Notes)

$$R_r = 31{,}200\left(\mu_{eff}\ \frac{NS}{\lambda^2}\right)^2 = 45\Omega \qquad \text{(3-54) Text}$$

[4] R. Pettengill, H. Garland, and J. Mendl, "Receiving antennas for miniature receivers," *IEEE Transactions on Antennas and Propagation*, Vol AP-26, pp. 528-530, July 1977.

From Eq. 11 above

$$L = 0.232\text{H} \quad \Rightarrow \quad \omega L = 2\pi \times 1.5 \times 10^6 \times 0.232 = 2.18 \text{ M}\Omega$$

$$D = 1.5; \quad e_r = \frac{R_r}{R_{ohmic} + R_r} = 0.82 \ (82\%)$$

$$G = e_r D = 1.23$$

pp. 107-111 **<u>Antennas in Communication Systems</u>**

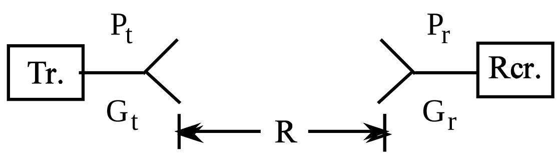
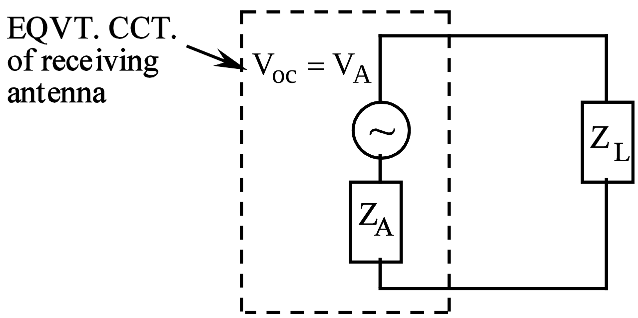


Fig. 4-4 (p. 107 Text). A communication link.



Equivalent circuit for the receiving system.

Maximum available power to the receiver (for $Z_L = Z_A^*$)

$$P_{Am} = \frac{1}{2}\left(\frac{V_A}{2R_A}\right)^2 R_A = \frac{V_A^2}{8R_A} = \frac{1}{8}\frac{\left|E^i\right|^2 (\Delta z)^2}{R_A} \qquad (1)$$

For an ideal (infinitesimal) dipole

$$V_A = E^i \, \Delta z \qquad (2)$$

Maximum effective aperture area = $A_{e,m} = \dfrac{P_{Am}}{S_{inc}} = \dfrac{3}{8\pi}\lambda^2 = 0.119\,\lambda^2$ (3)

$$S_{inc} = \left|E^i\right|^2 \big/ 2\eta \qquad (4)$$

$$D = 1.5 \text{ for an ideal dipole} \qquad (5)$$

(4-22)

$$D = \frac{4\pi}{\lambda^2} A_{e,m} = \frac{4\pi}{\lambda^2} \times \frac{3}{8\pi} \lambda^2 = 1.5 \qquad (6)$$

(4-23) Text

For a general antenna, therefore

$$G = \frac{4\pi}{\lambda^2} A_e \qquad (7)$$

$A_e = e_r A_{em}$ effective aperture area of an antenna (4-27)

p. 108 Text

Available power including also the antenna losses

$$P_A = S A_e \qquad \text{(4-26) Text}$$

$$S = G_t \frac{P_t}{4\pi R^2} \qquad (4\text{-}31)$$

$$P_r = S A_{er} = \left(\frac{G_t P_t}{4\pi R^2} \right) A_{er} = P_t \frac{G_t G_r \lambda^2}{(4\pi R)^2} \qquad (4\text{-}33)$$

or

$$P_r = P_t \frac{A_{et} A_{er}}{R^2 \lambda^2} \qquad \text{Friis transmission formula} \qquad (4\text{-}33)$$

We can also write Eq. (4-33) in dB-form as follows:

$$P_r(\text{dBm}) = P_t(\text{dBm}) + G_t(\text{dB}) + G_r(\text{dB}) - 20\log R(\text{km}) - 20\log f(\text{MHz}) - 32.44 \qquad (4\text{-}34)$$

Example 6:

For Ground Based TV Stations

Say, Channel 5 f = 76 - 82 MHz $f \cong 80$ MHz $\lambda = 3.75$ m

$P_{rad} \sim 5 - 10$ kW

say, 10 kW = 10^4 W (40 dBW)

G_t ~ 20 - 50 (factor)

say, $G_t = 30$ (factor) $\Rightarrow$ (14.77 dB ~ 15 dB)

EIRP = $G_t P_{rad} \rightarrow$ 55 dBW $\rightarrow 10^{5.5}$ W (85 dBm)

R_{max} ~ 20 - 30 miles ~ 50 km; since 1 mile = 1.6 km

say

$G_r \cong 7 \text{ dB} \cong 5$

$$A_{e,r} = \frac{\lambda^2}{4\pi} G_r = 5.6 \text{ m}^2$$

Using the logarithmic form of the Friis communication link formula Eq. (4-34)

$$\begin{aligned} P_r(\text{dBm}) &= 70 + 15 + 7 - 34.0 - 38.06 - 32.44 \\ &= -12.5 \text{ dBm} = 10^{-12.5} \text{mW} = 56.2\ \mu\text{W} \end{aligned}$$

Example 7:

Calculate the open-circuit voltage developed across an antenna of resistance R_A= 80 ohms for the above-calculated incident power density

$$S_{inc} = \frac{P_r}{A_e} = \frac{56.2}{5.6} = 10 \frac{\mu\text{W}}{\text{m}^2}$$

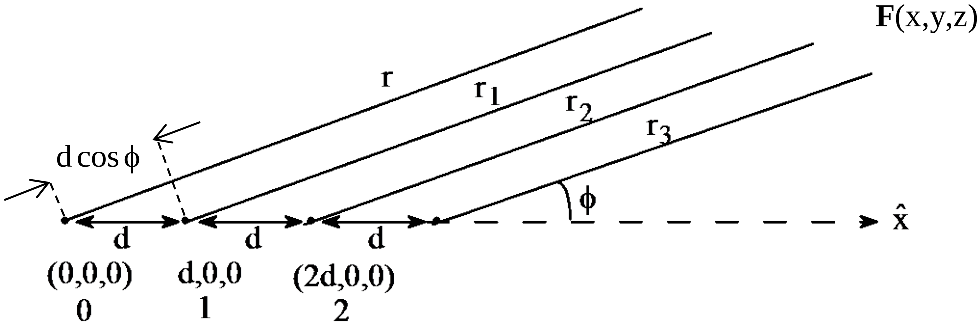
Assume R_A= 80Ω

$$\frac{V_{oc}^2}{8R_A} \rightarrow \frac{V_A^2}{8R_A} = \text{power picked up and delivered to a matched load} = S_{inc} A_e$$

$$\begin{aligned} V_{oc} &= \sqrt{8R_A S_{inc} A_{e,r}} = \sqrt{8 \times 80\ P_r} = \sqrt{640 \times 56.2 \times 10^{-6}} \\ &= 188.65 \text{ mV} \simeq 0.19\text{V} \end{aligned}$$

Chapter 8 -- Antenna Arrays (see pp. 271..... Text)

For a uniformly excited (UE), equally-spaced linear array (ESLA)



For N **identical** radiating elements (length, orientation, etc.) that are excited with identical magnitudes but progressively phase-shifted currents i.e.

$$I,\ I\,e^{-j\alpha},\ I\,e^{-2j\alpha}, \cdots I\,e^{-j(N-1)\alpha} \tag{1}$$

we can write the total electric field $\vec{E}_T$ as follows

$$\vec{E}_T = \vec{E}_o e^{-j\vec{\beta}\cdot\vec{r}} + \vec{E}_1 e^{j\vec{\beta}\cdot\vec{r}_1} + \cdots \vec{E}_{N-1}\, e^{-j\vec{\beta}\cdot\vec{r}_{N-1}} \tag{1}$$

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r}_1 &= (x-d)\hat{x} + y\hat{y} + z\hat{z} \end{aligned} \tag{2}$$

$$\vec{\beta} = \beta\left[\sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}\right] \tag{3}$$

From Eq. 1, we can write

$$\begin{aligned} \vec{E}_T &= \vec{E}_o e^{-j\vec{\beta}\cdot\vec{r}}\left[1 + e^{-j\alpha} e^{j\beta d \sin\theta\cos\phi} + e^{-2j\alpha} e^{2j\beta d\sin\theta\cos\phi} + \cdots\right] \\ &= E_o e^{-j\vec{\beta}\cdot\vec{r}} \sum_{n=0}^{N-1} e^{jn\psi} \end{aligned} \tag{4}$$

(3-16)

since

$$\vec{\beta}\cdot\left(\vec{r}_1 - \vec{r}\right) = -\beta d\sin\theta\cos\phi; \quad \vec{\beta}\cdot\left(\vec{r}_2 - \vec{r}\right) = -2\beta d\sin\theta\cos\phi \tag{6}$$

From Eq. 4

$$\vec{E}_T = \vec{E}_o \cdot AF$$

where

$$\text{Array Factor } AF = \sum_{n=0}^{N-1} e^{jn\psi} = \frac{1-e^{jN\psi}}{1-e^{j\psi}} \tag{7}$$

$$AF = e^{j(N-1)\psi/2} \frac{\sin(N\psi/2)}{\sin(\psi/2)} \tag{8}$$

(8-19) p. 279 Text

$$\text{Normalized AF} \quad f(\psi) = \frac{\sin(N\psi/2)}{N\sin(\psi/2)} \tag{9}$$

(8-22 Text)

UE, ESLA

where

$$\begin{aligned}\psi &= \left(\beta d_x \sin\theta\cos\phi - \alpha_x\right) \text{ for an } \hat{x}-\text{directed array}\\ &= \left(\beta d_y \sin\theta\sin\phi - \alpha_y\right) \text{ for a } \hat{y}-\text{directed array}\\ &= \left(\beta d_z \cos\theta - \alpha_z\right) \text{ for a } \hat{z}-\text{directed array} \quad \text{(see Eq. 3-19 Text)}\end{aligned} \tag{10}$$

$$\vec{E}_T = N\vec{E}_o f(\psi) = \vec{E}_o \frac{\sin(N\psi/2)}{\sin(\psi/2)}\ * \tag{11}$$

$$\vec{H}_T = \frac{\nabla \times \vec{E}_T}{j\omega\mu_o} = -\frac{j\beta}{j\omega\mu_o}\hat{r} \times \vec{E}_T \tag{12}$$

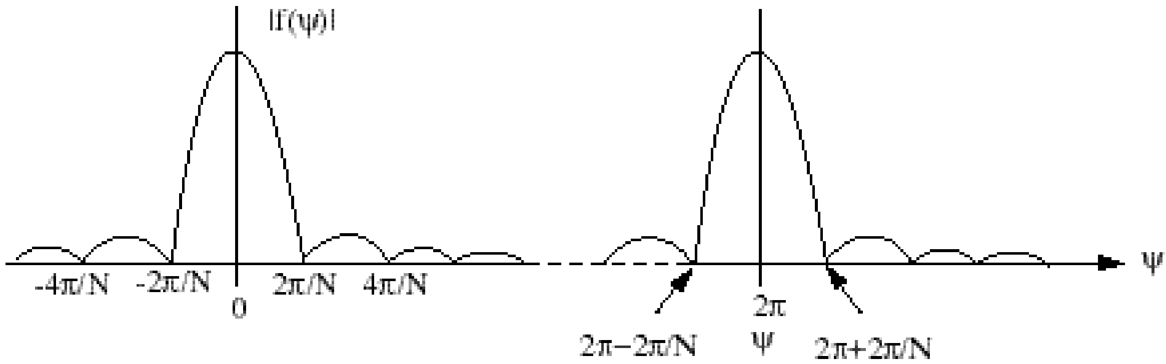
$$\vec{S} = \frac{1}{2}\text{Re}\left(\vec{E}_T \times \vec{H}_T^*\right) = \frac{\vec{E}_T \cdot \vec{E}_T^*}{2\eta}\hat{r} \tag{13}$$

* From Eq. 11, **for directions of max radiation** $\frac{\psi}{2} = 0, \pm\pi, \pm 2\pi, \cdots$

p. 280 **<u>Text</u>**

A number of trends can be seen by examining the normalized array factor $|f(\psi)|$.

1. <u>As N increases the main lobe narrows.</u> Peak for the main lobe occurs for $\psi = 0$ where $|f(\psi)| = 1$.



Plot of |f(ψ)| as a function of ψ.
Fig. (8-8) p. 280 Text.

For directions of zero (nulls of radiation)

Zero values of |f(ψ)| occur for

$$\frac{N\psi}{2} = \pm\,\pi, \pm\,2\pi, \cdots$$

i.e.

$$\psi = \pm\frac{2\pi}{N}, \quad \pm\frac{4\pi}{N}, \cdots \tag{14}$$

2. More than one major lobe will exist if it is possible to get values of $\psi = \pm\, 2\pi, \pm\, 4\pi$. The additional lobes are called <u>Grating Lobes.</u>

3. The minor lobes are of width $2\pi/N$ in the variable ψ and the major lobes (main and grating) are twice this width i.e. $4\pi/N$ in the variable ψ.

4. The side lobe peaks decrease relative to the major lobe as

$$1 : \frac{1}{N\sin\left(\frac{3\pi}{2N}\right)} \; : \; \frac{1}{N\sin\left(\frac{5\pi}{2N}\right)} \tag{16}$$

For large N, SLL decrease as

$$1 : \frac{2}{3\pi} : \frac{2}{5\pi} : \cdots \tag{17}$$

i.e.

$$0, \; 20\log\frac{2}{3\pi}, \; 20\log\frac{2}{5\pi}, \; \cdots$$

or

$$0, \; -13.46, \; -17.90, \; \cdots \quad \text{dB} \tag{18}$$

5. As N increases, there are more side lobes in one period of f(ψ). See also the text, Fig. 8-8, p. 280.

Case A. Broadside Arrays

All antennas excited in phase $\alpha = 0$.

From Eq. 10, for an antenna stretched along the z-axis

$$\psi = \beta d \cos\theta = 0, \quad \pm 2\pi, \quad \pm 4\pi \;\square \tag{19}$$

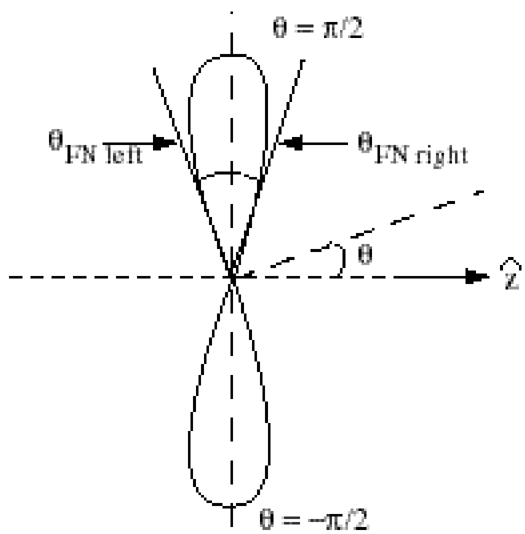
for major lobes

$$\theta = \pm\frac{\pi}{2}, \quad \cos^{-1}\left(\pm\frac{\lambda}{d}\right), \quad \cos^{-1}\left(\pm\frac{2\lambda}{d}\right)\cdots \tag{20}$$

Subcase 1

$d/\lambda < 1$ i.e. interelement spacing less than λ

Two and only two major lobes of radiation for $\theta = \pm\pi/2$ i.e. in **directions broadside** to the stretch of the array



For directions of first nulls (θ_{FN})

$$\beta d\cos\theta_{FN} = \pm\frac{2\pi}{N} \tag{21}$$

$$\theta_{FN} = \cos^{-1}\left(\pm\frac{\lambda}{Nd}\right) \tag{22}$$

$$\beta d\sin\left(\frac{\pi}{2} - \theta_{FN}\right) = \pm\frac{2\pi}{N} \tag{23}$$

$$BWFN = \left|\theta_{FN\,left} - \theta_{FN\,right}\right| \tag{24}$$

$$= \left|\cos^{-1}\left(-\frac{\lambda}{Nd}\right) - \cos^{-1}\left(\frac{\lambda}{Nd}\right)\right| \tag{25}$$

(8-31) p. 283 Text

$$= 2\sin^{-1}\left(\frac{\lambda}{Nd}\right) \cong \frac{2\lambda}{Nd}\,radians = 114.6° \frac{\lambda}{Nd} \tag{26}$$

(8-33)

for

$$Nd >> \lambda$$

Example 6:

$$d/\lambda = 0.5\,, \quad N = 8$$

From Eq. 23

$$\frac{\pi}{2} - \theta_{FN} = \pm \sin^{-1}\left(\frac{1}{4}\right) = \pm 14.5^{\circ} \tag{27}$$

$$BWFN = 29° \tag{28}$$

Angle for first-side lobe

$$\sin\left(\frac{N\psi}{2}\right) = \pm\ 1$$

from Eq. 11

$$\frac{N\psi}{2} = 4(\beta d \cos\theta) = \frac{8\pi d}{\lambda}\cos\theta = \pm\frac{3\pi}{2} \tag{29}$$

$$\theta = \cos^{-1}\left(\pm\frac{3}{8}\right) = \pm 68^{\circ}\,; \quad \pm 112^{\circ} \tag{30}$$

(-13.46 dB down relative to major lobe)

Subcase 2

$$1 \le \frac{d}{\lambda} \le 2$$

From Eq. 19, **for major lobes**

$$\psi = \beta d \cos\theta = 0, \quad \pm 2\pi, \quad \pm 4\pi, \ldots \tag{31}$$

$$\cos\theta = 0, \quad \pm\frac{\lambda}{d}, \quad \pm\frac{2\lambda}{d} \tag{31a}$$

This corresponds to six major lobes and a radiation pattern of the type shown on the next page.

Example 7:

$$d/\lambda = 1.5$$

From Eq. 31, **for major lobes**

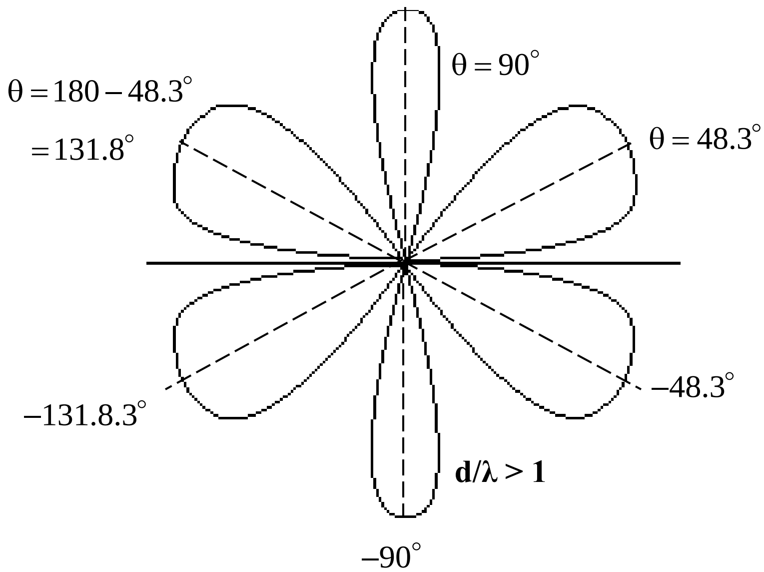
$$3\pi \cos\theta = 0, \quad \pm 2\pi, \quad \pm 4\pi \tag{32}$$

For angles of maximum radiation

$$\cos\theta = 0, \quad \pm 2/3, \quad \cancel{\pm 4/3} \tag{33}$$

$$\theta = \pm 90°; \quad \pm 48.2°, \quad \pm 131.80° \tag{34}$$

Six maxima of radiation



Angles for first nulls for each of these maxima are obtained from Eq. 21

$$3\pi \cos\theta_{FN} = \pm\frac{2\pi}{N}; \quad \pm 2\pi \pm \frac{2\pi}{N} \tag{35}$$

Subcase 3

$$d/\lambda = 1.0$$

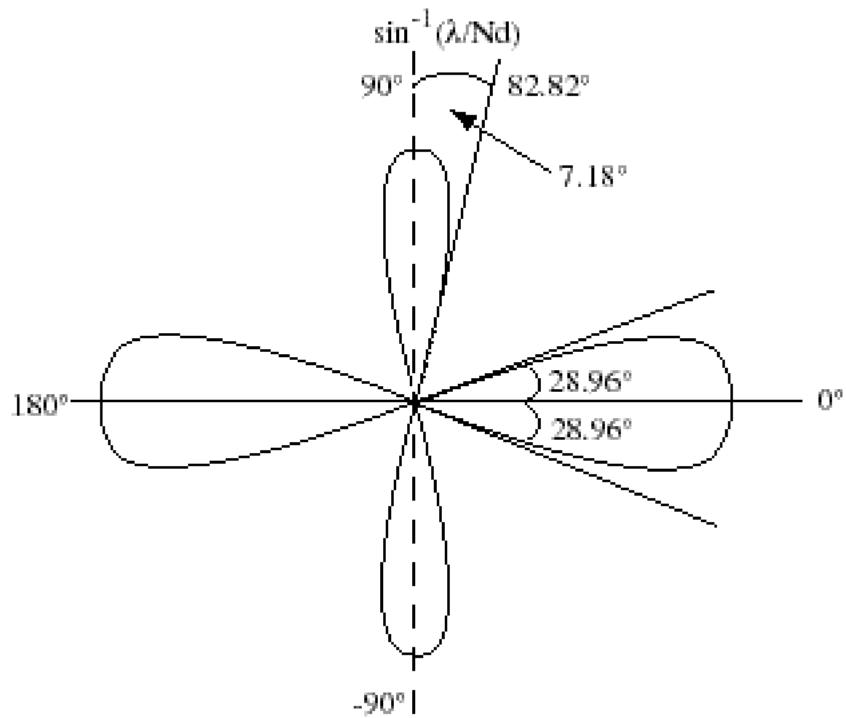
For this case, there are four maxima of radiation (major lobes); the two fatter lobes in the above figure coalesce into single modes with directions of maximum radiation θ = 0°, 180°.

From Eq. 31a

$$\cos\theta = 0, \quad \pm 1, \quad \cancel{\pm 2} \tag{36}$$

$$\theta = \pm 90^\circ,\quad 0^\circ,\quad 180^\circ \tag{37}$$

Four maxima of radiation



From Eq. 21, directions of first nulls are:

$$\psi = \frac{2\pi}{\lambda} d \cos\theta_{FN} = \pm\frac{2\pi}{N}; \qquad \pm 2\pi \pm \frac{2\pi}{N}, \cdots \tag{38}$$

$$\cos\theta_{FN} = \pm\frac{\lambda}{Nd}, \qquad \pm 1 \pm \frac{\lambda}{Nd}, \cdots \tag{38a}$$

Example 8:

$$N = 8, \quad d/\lambda = 1.0$$

From Eq. 38a

$$\cos\theta_{FN} = \pm\frac{1}{8}, \quad \pm 1 \pm \frac{1}{8} = \pm\frac{1}{8}, \quad \frac{7}{8}, \quad -\frac{7}{8} \tag{39}$$

$$\theta_{FN} = 82.82^\circ, \quad 97.18^\circ, \quad 28.96^\circ, \quad -28.96^\circ, \quad 151.04^\circ, \quad 208.96^\circ \tag{40}$$

BWFN = 14.36° **for major lobes along ± 90°**

= 57.92° **for major lobes along 0°, 180°**

p. 315 **Case B.** **<u>Electronically-Scannable (Steerable) Antennas -- Phased Array Antennas</u>**

The phase shift of currents (excitations) for adjacent antennas may be altered

$$\alpha \equiv -\beta d \cos\theta_o \qquad \text{(phase delay)} \tag{41}$$

From Eq. 11 for directions of maximum radiation (major lobes)

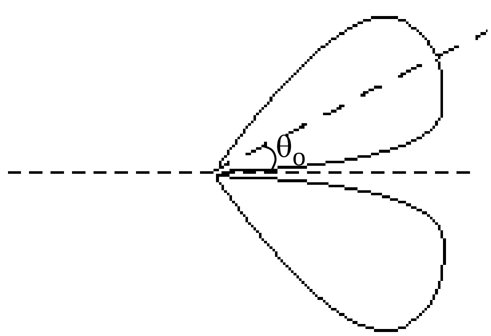
$$\frac{\psi}{2} = 0, \quad \pm\pi, \quad \pm 2\pi, \cdots \tag{42}$$

$$\psi = 0, \quad \pm 2\pi, \quad \pm 4\pi, \cdots \tag{43}$$

$$\beta d\left(\cos\theta - \cos\theta_o\right) = 0, \quad \pm 2\pi, \quad \pm 4\pi, \cdots \tag{44}$$

$$\cos\theta = \cos\theta_o, \quad \cos\theta_o \pm \frac{\lambda}{d}, \quad \cos\theta_o \pm 2\frac{\lambda}{d}, \cdots \tag{45}$$

For two and only two major lobes for $\theta = \pm\,\theta_o$, d/λ should be less than 0.5.



$$\theta_o = \cos^{-1}\left(-\frac{\alpha}{\beta d}\right)$$

Example 9:

$$N = 8, \quad d/\lambda = 0.3; \quad \alpha = -30°; \quad \theta_o = \cos^{-1}\left(\frac{\pi/6}{2\pi \times 0.3}\right) = \pm\, 73.9°$$

For

α variable from -30° to -75°

θ_o varies from ± 73.9° to ± 46°

For directions of zero radiation, from Eq. (14) on p. 25 of Class Notes,

$$\frac{N\psi}{2} = \pm\pi$$

$$\psi = \pm\frac{2\pi}{N}$$

$$\beta d\left(\cos\theta - \cos\theta_o\right) = \pm\frac{2\pi}{N}$$

$$\cos\theta_{FN} = \cos\theta_o \pm \frac{2\pi}{N\beta d}$$

$$\theta_{FN} = \cos^{-1}\left[\cos\theta_o \pm \frac{2\pi}{N\beta d}\right]$$

$$= \cos^{-1}\left[\cos\theta_o \pm \frac{\lambda}{Nd}\right] \qquad (45a)$$

Example 9 (continued): $d/\lambda = 0.3$, $N = 8$

$\alpha = -30° = -\pi/6$ $\qquad \theta_o = \pm 73.9°$

$$\theta_{FN} = \cos^{-1}\left[0.2773 \pm \frac{1}{2.4}\right] \quad (\tfrac{1}{2.4} = 0.4167)$$

$= 46.05°$; $98.01°$

BWFN = 51.96°

Example 9, Part B: Let us compare the antenna array of N = 8, d = 0.3λ for the following three conditions:

α		Direction of max radiation	BWFN
0	Broadside array	$\theta_o = \pm 90°$ from Eq. (20) on p. 27 of Class Notes	from Eq. (26) on p. 28 of Class Notes $BWFN = 2\sin^{-1}\left(\frac{\lambda}{Nd}\right)$ = 49.25°
-30° from Ex. 9 on this page	directions of max radiation $\theta_o = \pm 73.9°$	$\theta_o = \pm 73.9°$	BWFN = 51.96°
-108°	End fire antenna array $\alpha = -\beta d$	$\theta_o = 0°$	$BWFN = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$ = 108.6° See Eq. 52 on p. 36 of Class Notes

For a one-dimensional antenna array

The array factor of a one-dimensional antenna array from Eq. (8) of Class Notes p. 25 is as follows:

$$|AF| = \left|\frac{\sin(N\psi/2)}{\sin\psi/2}\right| \tag{1}$$

Where ψ is given by Eq. (1) on p. 25 of Class Notes.

From Eq. (1) here, for directions of max radiation

$$\psi = 0, \pm 2\pi, \ldots$$

For directions of zero radiation or nulls of radiation

$$\frac{N\psi}{2} = \pm\pi, \pm 2\pi, \ldots$$

or $\psi = \pm 2\pi/N$ for <u>first nulls</u> of radiation.

<u>Table of general relationships for one-dimensional ẑ-directed phased array antennas</u>

α	directions of max. radiation principal lobe/s $\psi = 0$	directions of first nulls derived on p. 32	BWFN
0	$\theta_o = \pm 90°$ broadside array	$\cos^{-1}\left[\cos\theta_o \pm \frac{\lambda}{Nd}\right]$ $= \cos^{-1}\left[\pm\frac{\lambda}{Nd}\right]$ see Eq. (22) on p. 27 of Class Notes	$2\sin^{-1}\left(\frac{\lambda}{Nd}\right)$ see Eq. (26) on p. 28 of Class Notes
α	$\theta_o = \cos^{-1}\left(-\frac{\alpha}{\beta d}\right)$ see p. 31 of Class Notes	$\cos^{-1}\left[\cos\theta_o \pm \frac{\lambda}{Nd}\right]$ $= \cos^{-1}\left[-\frac{\alpha}{\beta d} \pm \frac{\lambda}{Nd}\right]$ see Eq. 45a on p. 32 of Class Notes	calculate θ_{FN1}, θ_{FN2} BWFN = $\theta_{FN2} - \theta_{FN1}$
$\alpha = -\beta d$	$\theta_o = \cos^{-1}(1)$ $= 0°$ End fire array	$\cos^{-1}\left[1 - \frac{\lambda}{Nd}\right]$	$2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$ $= 4\sin^{-1}\left(\sqrt{\frac{\lambda}{2Nd}}\right)$ see Eq. 52 on p. 36 of Class Notes

Case C. End Fire Arrays

From the previous section, we can see that in order to get a single major lobe for $\theta_o = 0°$ i.e. along the line or stretch of the array, we need

$$\alpha = -\beta d \quad \text{and} \quad d/\lambda < 0.5 \tag{47}$$

For this case, the two major lobes on the previous page coalesce into one major lobe in the end fire direction.

Example 10:

$$N = 20\,, \quad d/\lambda = 0.4$$

$$\alpha = -\beta d = -\frac{2\pi d}{\lambda} = -144^{\circ} \tag{48}$$

For directions of first nulls from Eq. 14

$$\psi = \beta d\left(\cos\theta_{FN} - 1\right) = \pm\frac{2\pi}{N} \tag{49}$$

$$\cos\theta_{FN} = 1 \pm \frac{\lambda}{Nd} = 1 \pm \frac{1}{8} = \cancel{\frac{9}{8}},\ \frac{7}{8} \tag{49a}$$

$$\theta_{FN} = \pm\cos^{-1}\left(\frac{7}{8}\right) = \pm\, 28.96^{\circ} \tag{50}$$

$$\text{BWFN} = 2\theta_{FN} = 57.92^{\circ} \tag{50a}$$

It is interesting to note that for a given stretch of the array (N-1)d or approximately Nd, **BWFN is smallest for broadside arrays, intermediate for phased arrays and broadest (largest) for end fire arrays.**

Example 11:

For N = 20, d = 0.4 broadside array ($\alpha = 0^{\circ}$)

$$\text{BWFN} = 2\sin^{-1}\left(\frac{\lambda}{Nd}\right) = 2\sin^{-1}\left(\frac{1}{8}\right) = 14.36^{\circ}$$

as compared to 57.92° for an end fire array.

Example 10 (continued): N = 20 ; d/λ = 0.4

For **<u>Hanson-Woodyard end fire array</u> (p. 285 Text)**

$$\alpha = -\left(\beta d + \frac{\pi}{N}\right) \quad \text{(8-37)} \quad ; \quad d < \frac{\lambda}{2}\left(1 - \frac{1}{20}\right) \quad \text{(8-38a)}$$

$$= -\left(144° + \frac{180°}{20}\right) \qquad\qquad d < 0.475\,\lambda$$

$$= -153°$$

For directions of first nulls (from Eqs. 10, 14 on pp. 25, 26 of Class Notes)

$$\psi = \beta d_z \cos\theta - \alpha_z = 144° \cos\theta_{FN} - 153° = \pm\frac{2\pi}{N} = \pm\frac{2\pi}{20} = \pm 18°$$

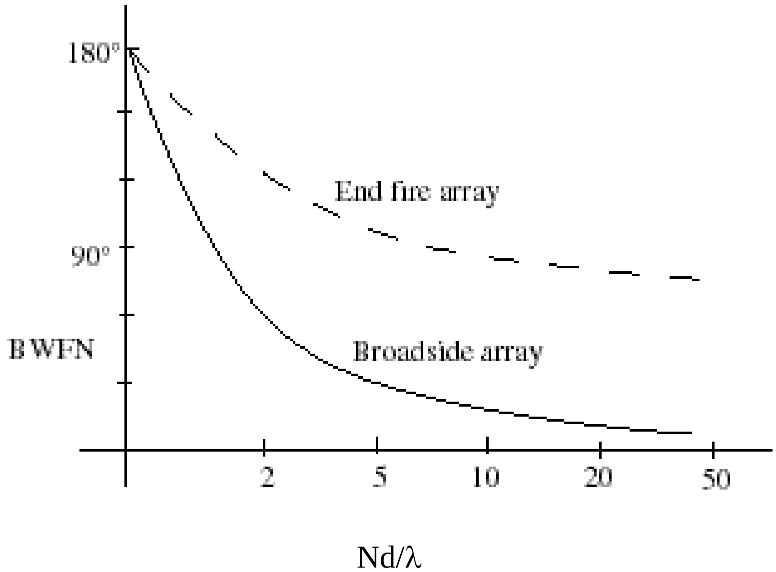
$$\cos\theta_{FN} = \frac{153° \pm 18°}{144°} = \frac{171°}{144°} \;\not{,}\; \frac{135°}{144°} = 0.9375$$

rather than 7/8 or 0.875 in Eq. 49a

$$\theta_{FN} = \pm\cos^{-1}(0.9375)$$

$$= \pm 20.36°$$

BWFN = 40.72° rather than 57.92° for an ordinary end fire array (see Eq. 50a on p. 34 of Class Notes)



BWFN for antenna arrays

$$\text{BWFN} = 2\sin^{-1}\left(\frac{\lambda}{Nd}\right) \longrightarrow \textbf{Broadside array} \qquad \textbf{(51)}$$

$$= 2\cos^{-1}\left(1-\frac{\lambda}{Nd}\right) \longrightarrow \textbf{End fire arrays} \qquad \textbf{(52)}$$

$$= 4\sin^{-1}\left(\sqrt{\frac{\lambda}{2Nd}}\right) \nearrow$$

Example 11-1: $\frac{Nd}{\lambda} = 5.0$

$\text{BWFN} = 23.07^\circ$ for a broadside array

$\text{BWFN} = 73.74^\circ$ for an end fire array

p. 293 **<u>Directivity of a Uniformally Excited, Equally Spaced Antenna Array</u>**

From Eq. 13 we can write

$$\vec{S} = \frac{\vec{E}_T\cdot\vec{E}_T^*}{2\eta}\hat{r} = \frac{|E_o|^2}{2\eta}\overbrace{\left|\frac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}}\right|^2}^{|AF|^2}\hat{r} \qquad (53)$$

$$\vec{S}_{max} = S_o\big|_{max}\,|AF|^2_{max} \Rightarrow N^2\,S_o\big|_{max} \qquad (54)$$

Power radiated by the antenna array = $\frac{1}{2}I_A^2\left(R_{A0}+R_{A1}+\cdots R_{N-1}\right)$

$$= \frac{1}{2}|I_A|^2\sum_{i=0}^{N-1} R_{Ai} \qquad (55)$$

In general
$$D\big|_{array} = N^2 D_o \frac{R_{A0}}{\sum_{i=0}^{N-1} R_{Ai}} \qquad (55a)$$

where D_o and R_{A0} pertain to an isolated element of the antenna array.

Ignoring Mutual Impedance Effects

$$R_{A0} = R_{A1}, \ldots = R_{N-1}$$

$$\text{Power radiated by the antenna array} = \frac{1}{2}\left|I_A\right|^2 R_{A0}\, N = N\, P_o \qquad (56)$$

where P_o is the power radiated by the zeroth element

$$D = \frac{S_{max}}{\frac{N\,P_o}{4\pi r^2}} = D_o \cdot \frac{AF\big|^2_{max}}{N} = D_o N \qquad (57)$$

where D_o is the directivity of each of the antenna elements.

Example 12:

Calculate the directivity of an antenna array of 20 half wavelength ($L = \lambda/2$) dipoles that are fed in phase and consequently radiate in broadside directions. Neglect the mutual impedance effects for this problem.

Solution:

$$D = \frac{AF\big|^2_{max}}{N} = ND_o = 20 \times 1.64 = 32.8$$

Example 13:

a. Calculate the directivity/gain of an array of 30 vertical monopoles above ground each of length $H = L/2 = 0.35\ \lambda$ that are spaced a distance $d = 0.2\lambda$ from each other.

b. Calculate the relative phase difference between monopoles if the major lobe of radiation is to be in the end fire direction assuming an ordinary end fire array.

c. Calculate the BWFN for this array.

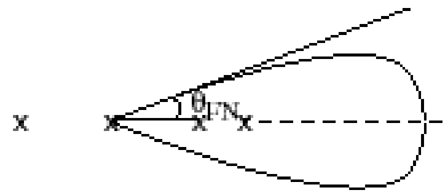
Solution:

a. $D = N\, D_o = 30 \times 3.636 = 109.08$

b. From Eq. 47

$$\alpha = -\beta d = -\frac{2\pi d}{\lambda} = -72^{\circ}$$

Each of the successive elements should be fed with a current that is lagging in phase by 72° from the previous element.



c. From Eq. 49a,

$$\cos\theta_{FN} = 1 - \frac{\lambda}{Nd} = 1 - \frac{\lambda}{6\lambda} = 1 - \frac{1}{6} = \frac{5}{6}$$

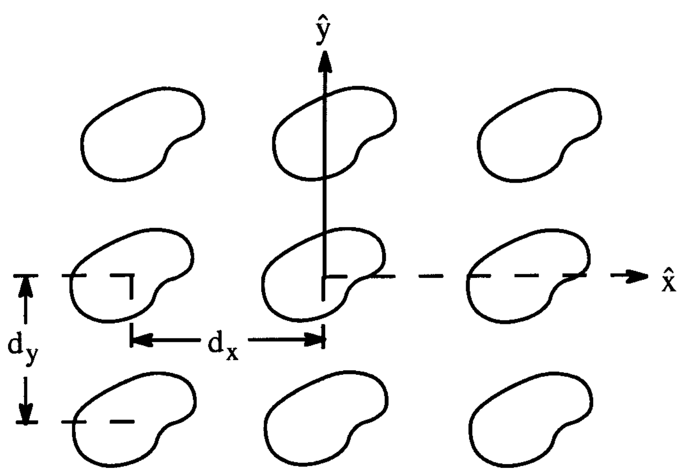
$$\text{BWFN} = 2\cos^{-1}(5/6) = 67.11°$$

2-D and 3-D Uniformly Excited, Equally-Spaced Antenna Arrays

N_x: No. of antennas in x-direction

N_y: No. of antennas in y-direction

N_z: No. of antennas in z-direction



A 2-D array of identical elements

Neglecting phase terms

$$\underline{E}_T = \underline{E}_{①} |AF|_x |AF|_y |AF|_z$$

$$|AF|_x = \frac{\sin\left\{\frac{N_x}{2}\left(\beta d_x \sin\theta\cos\phi + \alpha_x\right)\right\}}{\sin\left\{\frac{1}{2}\left(\beta d_x \sin\theta\cos\phi + \alpha_x\right)\right\}}$$

$$|AF|_y = \frac{\sin\left\{\frac{N_y}{2}\left(\beta d_y \sin\theta\sin\phi + \alpha_y\right)\right\}}{\sin\left\{\frac{1}{2}\left(\beta d_y \sin\theta\sin\phi + \alpha_y\right)\right\}}$$

$$|AF|_z = \frac{\sin\left\{\frac{N_z}{2}\left(\beta d_z \cos\theta + \alpha_z\right)\right\}}{\sin\left\{\frac{1}{2}\left(\beta d_z \cos\theta + \alpha_z\right)\right\}}$$

As always

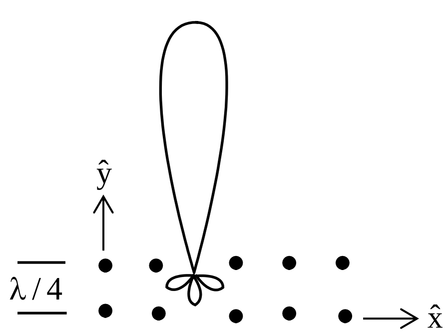
$$\vec{H}_T = \frac{\nabla \times \vec{E}_T}{-j\omega\mu_o} = \frac{-j\beta\hat{r} \times \vec{E}_T}{-j\omega\mu_o} = \frac{\hat{r} \times \vec{E}_T}{\eta}$$

$$\vec{S}_T = \vec{S}_{①} |AF|_x^2 |AF|_y^2 |AF|_z^2$$

where $\vec{S}_{①}$ is the radiated power density due to one of the elements. These arrays are also called <u>mattress Arrays.</u>

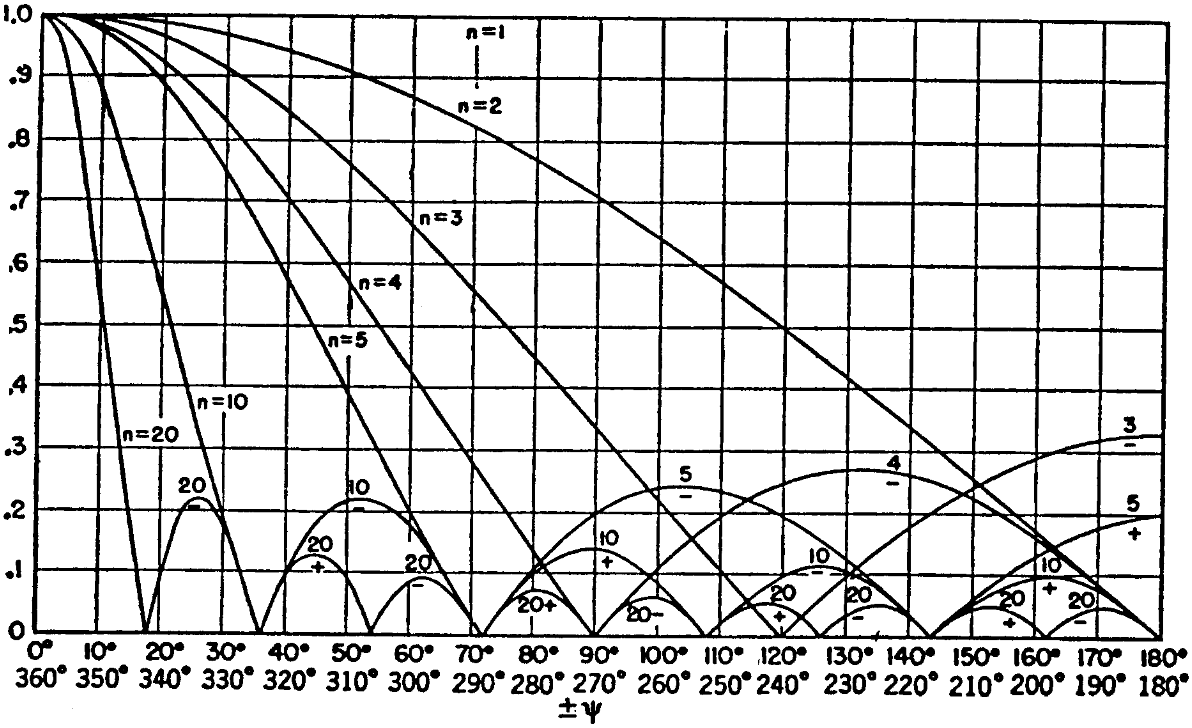
Example 14: A Unidirectional Broadside Array

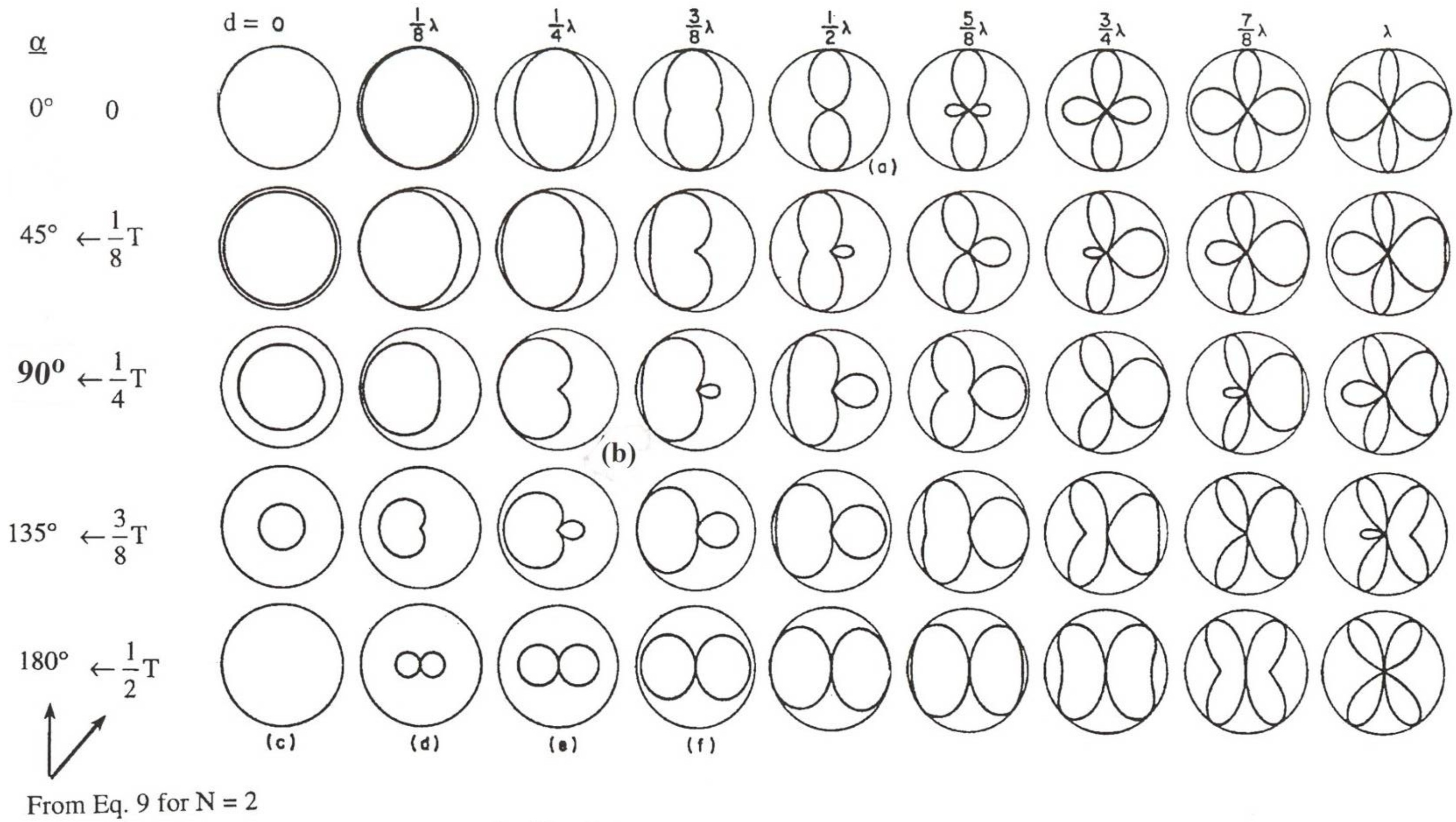
In order to obtain a unidirectional broadside array, we can use a 2-D antenna array of $N_z = 1$, $N_y = 2$, N_x which can be an arbitrary number. By using a back row of antennas that are placed with $d_y = \lambda/4$ and $\alpha_y = 90°$, we can obtain an antenna pattern as shown.



Universal field-pattern chart for arrays of various numbers n of isotropic point sources of equal amplitude and spacing

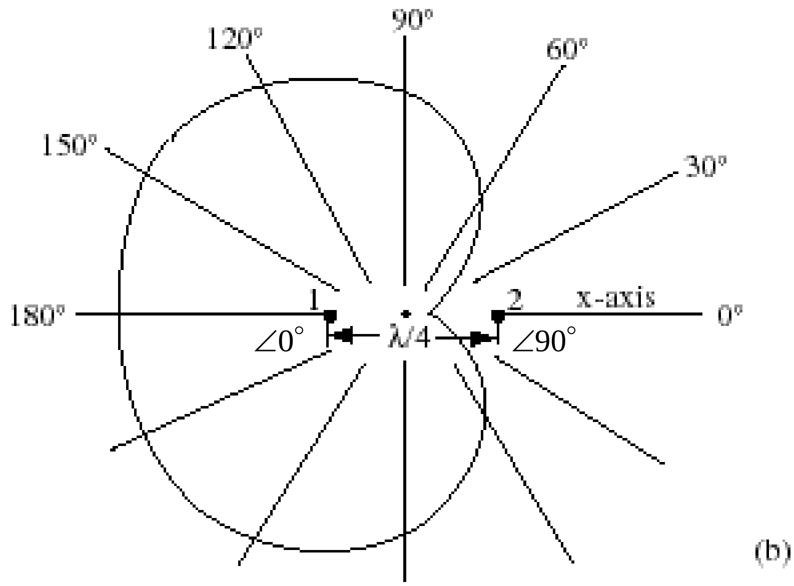
|Array factor| or $|f(\psi)|$





$$\frac{\sin(N\psi/2)}{N\sin(\psi/2)} = \cos\psi/2 \qquad \psi = \beta d\cos\boldsymbol{\theta} + \alpha$$

Directive amplitude diagrams for **an array of two antennas** driven by currents of equal amplitude. Separation in wavelengths (λ) along the top. Phase difference in periods (T) at left.



An enlarged version of Fig. (b) from previous page.

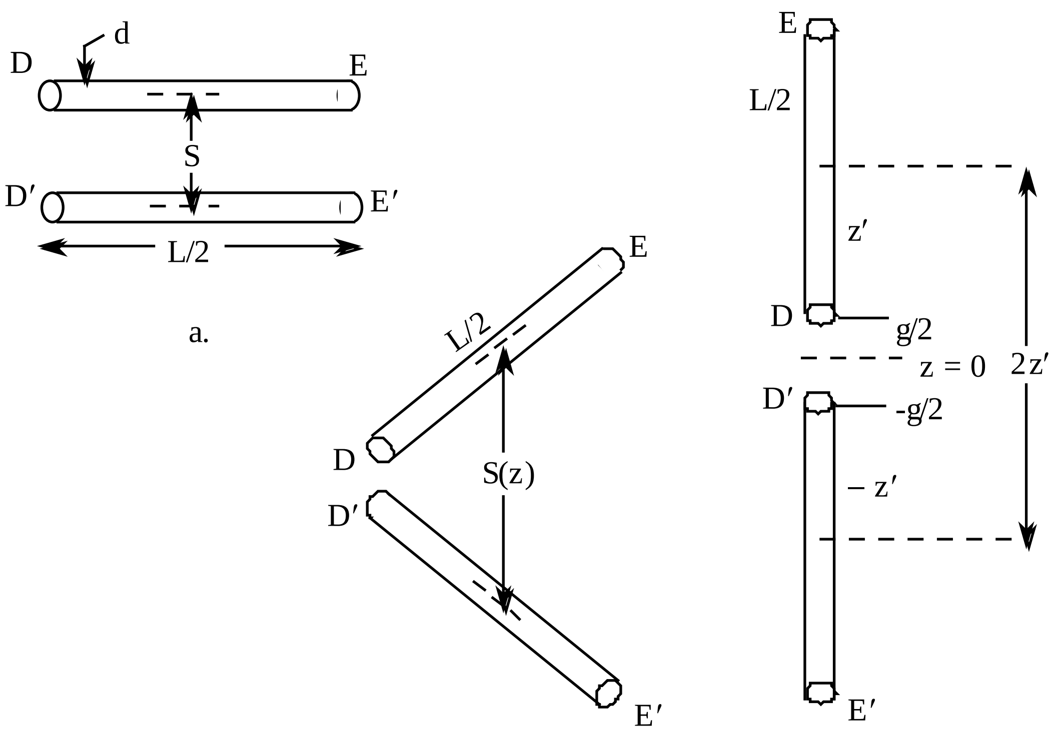
$\alpha = 90°$; $d = \lambda/4$

Note a broad unidirectional (cardiod type) pattern possible with this arrangement.

Reactance of Linear Dipoles

We have previously calculated $R_{in} = R_a + R_{ohmic}$ for linear dipole or monopole antennas. We need to know the input reactance X_{in} or X_a in order to design matching networks to match power in or out of the antenna.

Like the current distribution on a linear dipole, the input reactance can be written as though a two-wire line of length L/2 had been opened up as shown in the following:



a. b. c.

For a two-wire line of Fig. a, each of diameter d,

- $$Z_o = \frac{120}{\sqrt{\varepsilon_r}} \ell n\left(\frac{2S}{d}\right) \tag{1}$$

For the opened-up line of Fig. b, we can define an average characteristic impedance $\overline{Z}_o$

- $$\overline{Z}_o = \frac{1}{L/2}\int_0^{L/2} Z_o(z)\, dz \tag{2}$$

For the completely opened-up transmission line of Fig. c, we can define

$$\overline{Z}_{oa} = \frac{2}{L}\int_{g/2}^{g/2+L/2} \frac{120}{\sqrt{\varepsilon_r}}\,\ell n\left(\frac{4z'}{d}\right)dz'$$
$$= \frac{120}{\sqrt{\varepsilon_r}}\left\{\ell n\left(\frac{2L}{d}\right)-1\right\} \tag{3}$$

The reactance $Z_{DD'}$ of an open-circuited transmission line of length L/2 can be written from Transmission Line Theory

$$Z_{DD'} = Z_{in} = -jZ_{oa}\cot\left(\beta\frac{L}{2}\right) \tag{4}$$

Combining Eq. 3 and 4, we can write

$$Z_{DD'} = jX_{in} = -j\,\overline{Z}_{oa}\cot\left(\frac{\beta L'}{2}\right) \tag{5}$$

where $L' \cong (1.02-1.10)L$ is the effective "electrical" length of the antenna.

<u>Reactance of Linear Monopoles Above Ground</u>

We have previously shown that

$$R_{in}\big|_{monopole} = \frac{1}{2}R_{in}\big|_{dipole} \tag{6}$$

Similarly,

$$\overline{Z}_{oa}\big|_{monopole} = \frac{1}{2}Z_{oa}\big|_{dipole} = 60\left\{\ell n\left(\frac{2L}{d}\right)-1\right\} \tag{7}$$

From Eq. 5

$$X_{in}\big|_{monopole} = \frac{1}{2}X_{in}\big|_{dipole} \tag{8}$$

Example 15:

Calculate the feed point impedances $R_{in} + jX_{in}$ for linear dipoles of length (a) L = 0.5λ (half wave dipole) and (b) L = 0.3λ. Assume that the antenna wire is No. 19 AWG (d = 9.12×10^{-4} m from Table B.2, p. 623) and frequency f = 30 MHz. Take copper as the material for the antenna.

a. From the table on driving point resistance, p. 7 of Class Notes

$$R_{ri}\Big|_{L/2\lambda=0.25} = 2\times 36.56 = 73.12\Omega \tag{9}$$

From p.13 of Class Notes, Eq. 9

$$R_{ohmic} = \frac{R_s}{\pi a}\frac{1}{\sin^2\left(\frac{\pi}{2}\right)}\left[\frac{L}{4} - \frac{\sin(\beta L)}{4\beta}\right] \tag{10}$$

$$\sigma_{copper} = 5.8 \times 10^7 \text{ S/m}$$

$$R_s = \frac{1}{\sigma\delta} = 2.61\times 10^{-4}\sqrt{f_{MHz}} \qquad \text{for copper}$$

$$R_s = 2.6\times 10^{-4}\sqrt{30} = 14.4\times 10^{-4}\Omega \qquad @ \quad f = 30 \text{ MHz} \tag{11}$$

$$L = \frac{\lambda}{2} = 5 \text{ m}$$

$$R_{ohmic} = \frac{14.4\times 10^{-4}}{\pi\times 4.06\times 10^{-4}}\,\frac{5}{4} = 1.411\Omega \tag{12}$$

$$R_{in} = R_{ri} + R_{ohmic} = 74.53\Omega \tag{13}$$

From Eq. 3 on p. 44 of Class Notes

$$Z_{oa} = 120\left\{\ell n\left(\frac{10}{9.25\times 10^{-4}}\right) - 1\right\} = 996.3\Omega \tag{14}$$

Taking $L' \cong 1.02\,L = 0.51\lambda$

$$\cot\left(\frac{2\pi}{\lambda}\times\frac{0.510\lambda}{2}\right) = -0.0314$$

From Eq. 5 on p. 44 of Class Notes

$$jX_{in} = -j\,996.3\cot\left(\frac{\beta L'}{2}\right) = +j\,31.3\Omega \tag{15}$$

$$Z_{in} = R_{in} + jX_{in} = 74.53 + j31.3\Omega \tag{16}$$

Note that if we had constructed a slightly shorter, say L = 0.49λ dipole

$$L' = 0.49 \times 1.02\lambda = 0.5\lambda$$

$$jX_{in} = -j\,996.3 \cot \beta L' \Rightarrow 0$$

$$Z_{in} = R_{ri} + R_{ohmic} + j0 = 69.46 + 1.41 + j0 \cong 71 + j0\Omega \qquad (17)$$

b. You can solve for the numbers for part b of the problem following the procedure indicated above.

Example 16:

Feedpoint impedance for a linear monopole of length L/2 = 0.25λ.

Solution:

From Eq. 16

$$Z_{in}\Big|_{monopole} = \frac{1}{2} Z_{in}\Big|_{dipole} = 37.27 + j15.65\Omega$$

Examples on Calculation of Im (Z_A) or Reactance of Antennas

Example 17: (See also Fig. 6-6, p. 157 Text)

L/λ = 0.4; wire radius a = 0.0005λ (same as in Fig. 6-6, p. 157 Text). Assume $L' = 1.04\,L$.

From Eq. 3 on p. 44 of Class Notes

$$\overline{Z}_{oa} = 120\left\{\ell n\left(\frac{0.8\lambda}{0.001\lambda}\right) - 1\right\} = 682.15\Omega$$

From Eq. 5 on p. 44 of Class Notes

$$Z_{DD'} = -j\,\overline{Z}_{oa} \cot\left(\frac{\pi L'}{\lambda}\right) = -j682.15 \cot (0.4\pi \times 1.04) = -j184.3\Omega$$

taking $L' = 1.04L$ (from Table 6-2 on p. 159 Text). From the graph in Fig. 6-6, p. 157 Text

$$\mathrm{Im}\left(Z_A\right) = -j180\Omega$$

Example 18:

L/λ = 0.3; wire radius a = 0.0014λ (one of the wire radii on p. 8 of Class Notes).

From Eq. 3, p. 44 of Class Notes

$$\overline{Z}_{oa} = 120\left\{\ell n\left(\frac{0.6\lambda}{0.0028\lambda}\right) - 1\right\} = 524.08\Omega$$

$$Z_{DD'} = -j524.08\cot\left(\frac{\pi}{\lambda}L'\right) = -j524.08\cot\left(0.3\pi \times \frac{L'}{L}\right) = -j351.4\Omega$$

taking $L'/L = 1.04$.

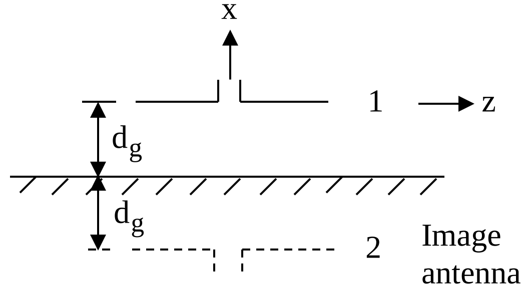
From graph on p. 8 of Class Notes

$$\text{Reactance } X_a = -2 \times 150 = -300\Omega$$

which is close.

Examples on Mutual Impedance Effects

Example 19: A half-wave dipole above ground



Distance to ground $d_g = \lambda/4$

$2d_g$ = distance to image antenna[2] = $\lambda/2$

From Fig. 8-25a, b, for $d/\lambda = 0.5$

$$Z_{12} = -12.5 - j30; \qquad I_2 = I_1\angle 180^\circ = -I_1$$

$$V_1 = I_1 Z_{11} + I_2 Z_{12} = I_1\left[Z_{11} - Z_{12}\right]$$

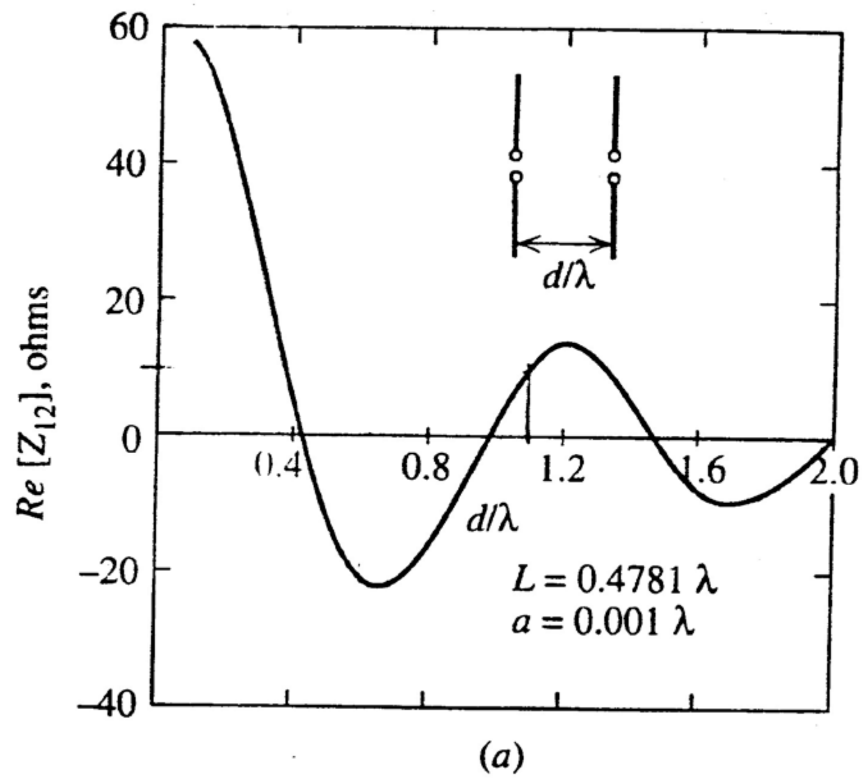
$$Z_1 = \frac{V_1}{I_1} = (73 + j42.5) - (-12.5 - j30)$$
$$= 85.5 + j72.5$$

Feedpoint impedance of the half-wave dipole placed at a distance of $\lambda/4$ from the ground = $85.5 + j72.5\Omega$ rather than $73 + j42.5\Omega$.

Radiation Pattern

We can consider the above situation as a 2-element (N = 2) antenna array in the x direction and write

$$\vec{E}_T = \vec{E}_{①}\,AF$$



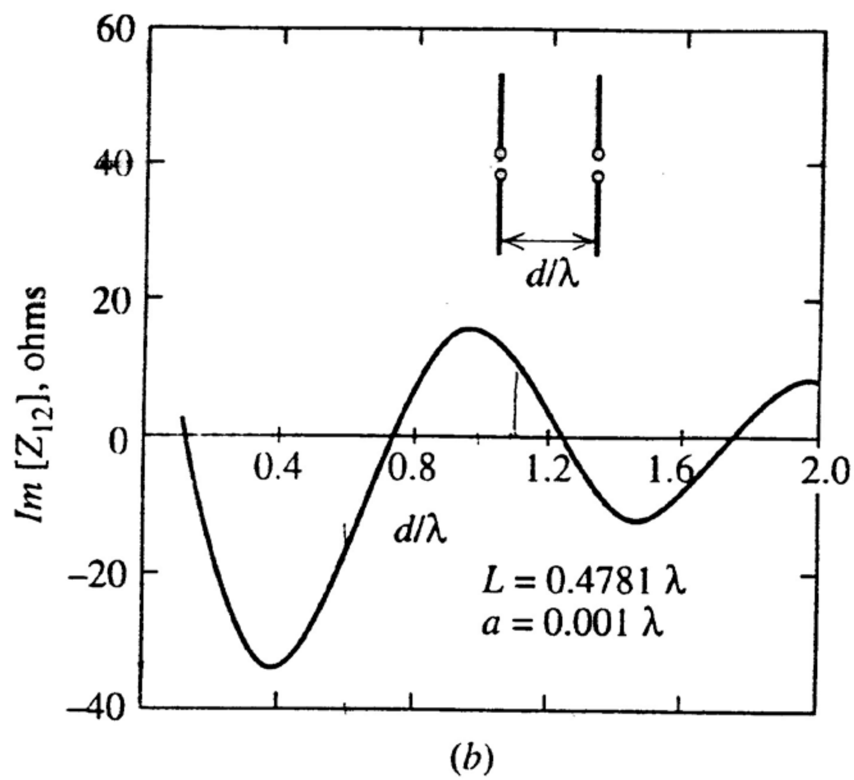


Figure 8-25 **The mutual impedance between two resonant parallel dipoles as a function** of their spacing relative to a wavelength. (a) The real part. (b) The imaginary part.

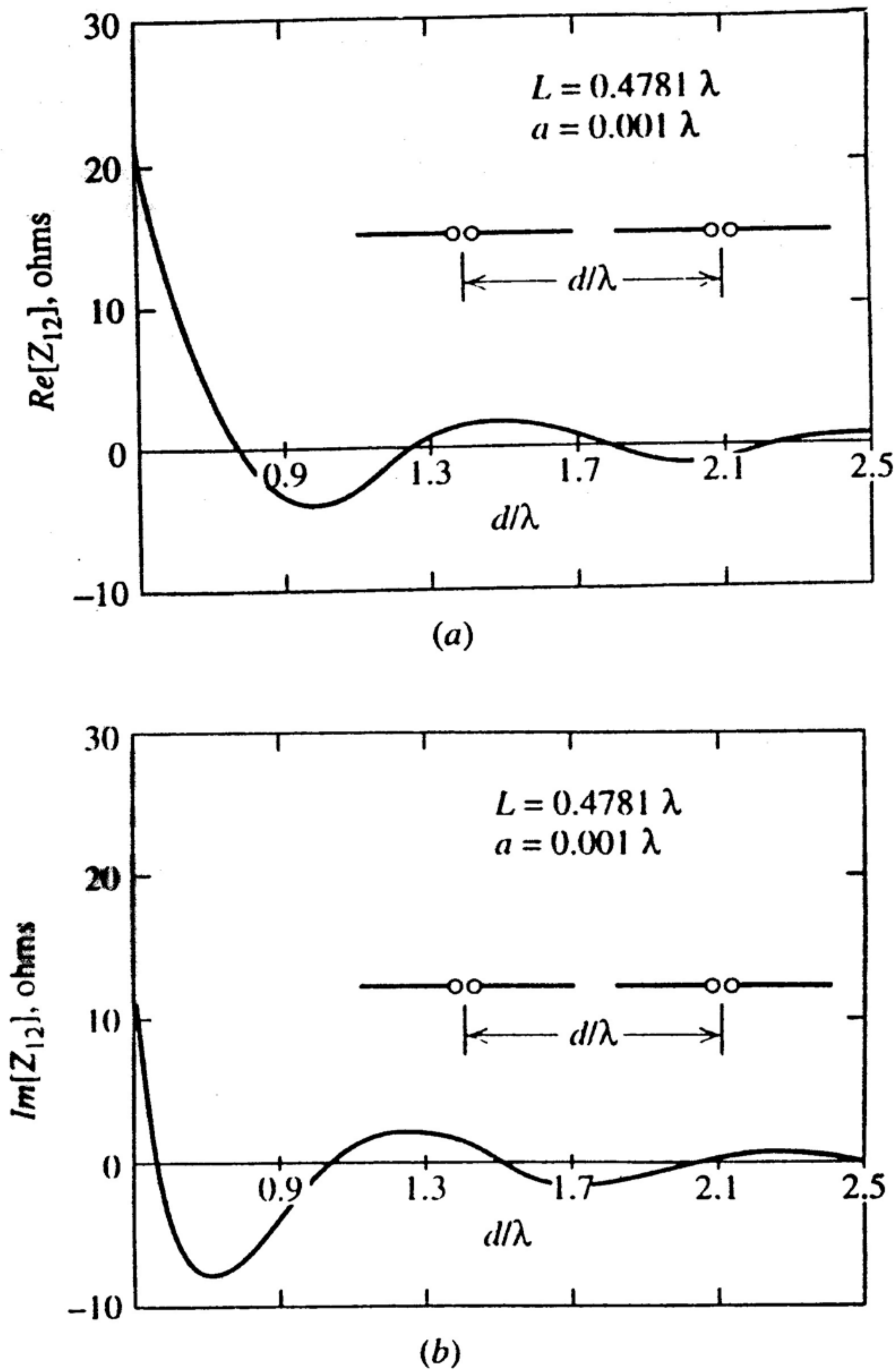


Figure 8-26 **The mutual impedance between two resonant collinear dipoles as a function** of spacing relative to a wavelength. (a) The real part. (b) The imaginary part.

$$AF = \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

where

$$\psi = \beta d_x \sin\theta\cos\phi + \alpha_x = 2\beta d_g \sin\theta\cos\phi + \pi$$
$$= \pi\sin\theta\cos\phi + \pi$$

From pp. 36-37 of Class Notes, Eqs. (54)-(57)

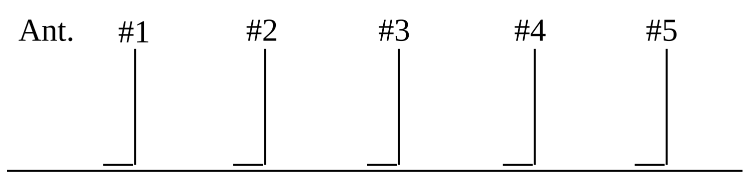
$$D = G = D_o \left|AF\right|^2_{max} \frac{R_{A,\text{isolated}}}{R_{A,\text{with ground effect}}} = 1.64 \times \cancel{N^2}\, 4 \times \frac{73}{85.5} = \boxed{5.60}$$

Without ground effect

$$D = G = 1.64$$

Example 20: A broadside array of five $\lambda/4$ monopoles ($\alpha = 0$)

$d = \lambda/2$



$d_{12} = \lambda/2$

$d_{13} = \lambda$ $\qquad d_{15} = 2\lambda$

$d_{14} = 3\lambda/2$ $\qquad I_1 = I_2 = I_3 = I_4 = I_5$ because it is a broadside array

$$Z_1 = \frac{V_1}{I_1} = Z_{11} + Z_{12} + Z_{13} + Z_{14} + Z_{15}$$
$$= \frac{1}{2}\left[(73 + j42.5) + (-12.5 - j30) + (4 + j18) + (-1.8 - j12) + (1 + j9)\right] = \frac{63.7 + j27.5}{2}$$
$$= 31.85 + j13.75\Omega$$

monopoles

$$\boxed{Z_5 = Z_1 \text{ by symmetry}}$$

Same as Z_{12}; $d_{25} = 3\lambda/2$; $d_{24} = \lambda$

$$Z_2 = Z_{12} + Z_{22} + Z_{23} + Z_{24} + Z_{25}$$

$$= \frac{1}{2}\left[2(-12.5 - j30) + (73 + j42.5) + (4 + j18) + (-1.8 - j12)\right]$$

monopoles (arrow to the factor $\frac{1}{2}$)

$$= \frac{1}{2}\left[50.2 - j11.5\right] = 25.1 - j5.75\Omega$$

$$\boxed{Z_2 = Z_4 \text{ by symmetry}}$$

Same as Z_{23}; Same as Z_{13}

$$Z_3 = Z_{13} + Z_{23} + Z_{33} + Z_{34} + Z_{35}$$

$$= 2Z_{13} + 2Z_{23} + Z_{33}$$

$$= \frac{1}{2}\left[2(4 + j18) + 2(-12.5 - j30) + (73 + j42.5)\right]$$

monopoles (arrow to the factor $\frac{1}{2}$)

$$= \frac{56 + j18.5}{2} = \boxed{28 + j9.25}$$

Note that for each of the antennas, the input impedances are slightly different and each of these values are different than

$$\frac{73 + j42.5}{2} \quad \text{or} \quad 36.5 + j21.25\Omega$$

for an isolated λ/4 **monopole**.

<u>Directivity</u>

From Eq. (55a) on p. 37 of the Class Notes, <u>including mutual</u> impedance effects

$$D = D_o \left|AF\right|^2_{max} \frac{R_A\big|_{isolated}}{\sum_{i=1}^{5} R_{Ai}}$$

$$= 3.28 \times \cancel{N}^2\, 25 \times \frac{36.5}{\underbrace{2\times 31.85}_{Re(Z_1 + Z_5)} + \underbrace{2\times 25.1}_{Re(Z_2 + Z_4)} + \underbrace{28}_{Re(Z_3)}} = \boxed{21.1}$$

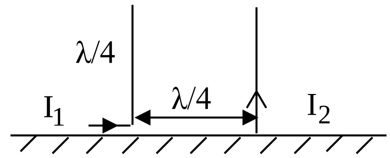
$D_o = 3.28 \Rightarrow 2\times 1.64$ for a single isolated λ/4 monopole above ground

Note that a directivity of 21.1 is higher than ND_o of 5 × 3.28 = 16.4 which would be obtained for this antenna array neglecting mutual impedance effects.

Inclusion of mutual impedance effects can often lead to an increased gain relative to the value had the mutual impedance effects been neglected.

Example 21: Two monopole antennas separated by $\lambda/4$. (<u>Note that the second antenna is grounded.</u>)

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$



$$0 = Z_{12}I_1 + Z_{22}I_2$$

$$I_2 = -\frac{Z_{12}}{Z_{22}} I_1 = -\frac{(36 - j25)/\not{2}}{(73 + j42)/\not{2}} I_1$$

$$= -\frac{43.8\, e^{-j34.87^\circ}}{84.22\, e^{+j29.91^\circ}} I_1 = 0.52\, e^{-j64.78^\circ} e^{j180^\circ} I_1$$

$$= \boxed{0.52\, e^{+j115.2^\circ} I_1}$$

For the driven antenna 1

$$Z_1 = \frac{V_1}{I_1} = Z_{11} + Z_{12}\frac{I_2}{I_1} \qquad (1)$$

From p. 307, Fig. 8-25 of the Text (see also p. 48 of the Class Notes)

$$Z_{12}\Big|_{d=\lambda/4} = \frac{36 - j25}{2} = 21.91\, e^{-j34.8^\circ}$$

(monopole, pointing to the 2 in the denominator)

From Eq. (1)

$$Z_1 = \frac{73 + j42}{2} + 0.52\, e^{j115.2^\circ}\left(21.91\, e^{-j34.87^\circ}\right)$$

$$= 38.4 + j32.2\Omega$$

<u>Calculate current I_1 for a transmitter power of 100 W</u>

Antenna 1 is the only antenna that is driven and is to be fed (current in antenna 2 is created by induction)

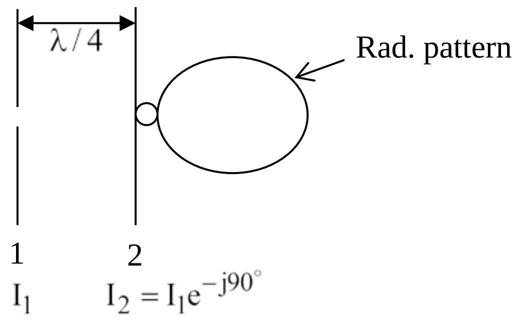
$$P_{rad} = 100W = \frac{1}{2} I_1^2\, R_{A1} = \frac{1}{2} I_1^2 \times 38.4$$

$$\boxed{I_1 = 2.28A}$$

Because of induced current (by mutual impedance effect)

$$I_2 = 0.52 \times I_1\, e^{j115.2^\circ} = \boxed{1.19\, e^{j115.2^\circ}}\text{A}$$

Example 22: Calculate the feedpoint impedances of two parallel antennae separated by a distance of λ/4 and fed with a phase shift α = -90º. Each of the antennas is a λ/2 dipole.



From Fig. 8-25, p. 307 Text (p. 48 of Class Notes)

$$Z_{11} = 73 + j42.5\,; \quad Z_{12} = 36 - j25$$

$$V_1 = I_1 Z_{11} + I_2 Z_{12} \Rightarrow Z_1 = \frac{V_1}{I_1} = \underset{\uparrow\ 73+j42.5}{Z_{11}} - j(36 - j25)$$

$$\underline{Z_1 = 48 + j6.5\Omega}$$

$$V_2 = I_1 Z_{12} + I_2 Z_{22} \Rightarrow Z_2 = \frac{V_2}{I_2} = j(36 - j25) + (73 + j42.5)$$

$$\underline{Z_2 = 98 + j78.5\Omega}$$

Power fed to Ant. 1 = $\frac{1}{2} I_1^2 \times 48 \qquad \rightarrow \quad 3.29\,\text{KW}$

Power fed to Ant. 2 = $\frac{1}{2} |I_1|^2 \times 98 \qquad \rightarrow \quad 6.71\,\text{KW}$

Total power = $\frac{1}{2} I_1^2 \left(R_{A1} + R_{A2}\right) = 10\,\text{KW}$

$$|I_1| = 11.7\text{ A}$$

$$G = N^2 G_1 \frac{R_{A,isolated}}{\sum_{i=1}^{2} R_{Ai}} = 4 \times 1.64 \times \frac{73}{146}$$

$$= 3.28$$

Methods of Matching Power to the Antennas

A. Transmission Line Matching Method

Example 23:

Match an antenna of impedance Z_a = 10 - j300Ω to a twin-wire line of characteristic impedance Z_o = 300Ω using (a) series elements and (b) a shunt element. Take f = 30 MHz

$$z_a = \frac{Z_a}{Z_o} = \frac{10 - j300}{300} = 0.033 - j1$$

This normalized impedance is shown as point A on the Smith Chart on page 55 of Class Notes. If the antenna is not matched

$$\text{Voltage reflection coefficient } \rho = \frac{Z_a - Z_o}{Z_a - Zo} = 0.9672\angle 270^{\circ} \tag{1}$$

$$\text{VSWR} = \frac{1+|\rho|}{1-|\rho|} = 60.0 \tag{2}$$

$$\text{Power reflection coefficient} = \frac{P_r}{P_{inc}} = |\rho|^2 = 0.9354 \tag{3}$$

i.e. 93.54% of the input power P_{inc} is reflected and only 6.46% of the transmitter power is radiated -- a truly poor situation!

Approach A: Use of series elements to match the antenna

From point A, we move on the transmission line circle C to point B on p. 55 -- Smith Chart, which corresponds to the intersection with real part z_B of 1.0 circle.

$$\text{Length } \ AB = (0.231 + 0.125)\lambda = 0.356\lambda \Rightarrow 3.56\text{m} \tag{4}$$

$$Z_{BB'} = z_B Z_o = (1 + j8)300 = 300 + j2400\Omega \tag{5}$$

As shown in Fig. 2, we can compensate for j2400Ω by using two capacitors as shown each of reactance

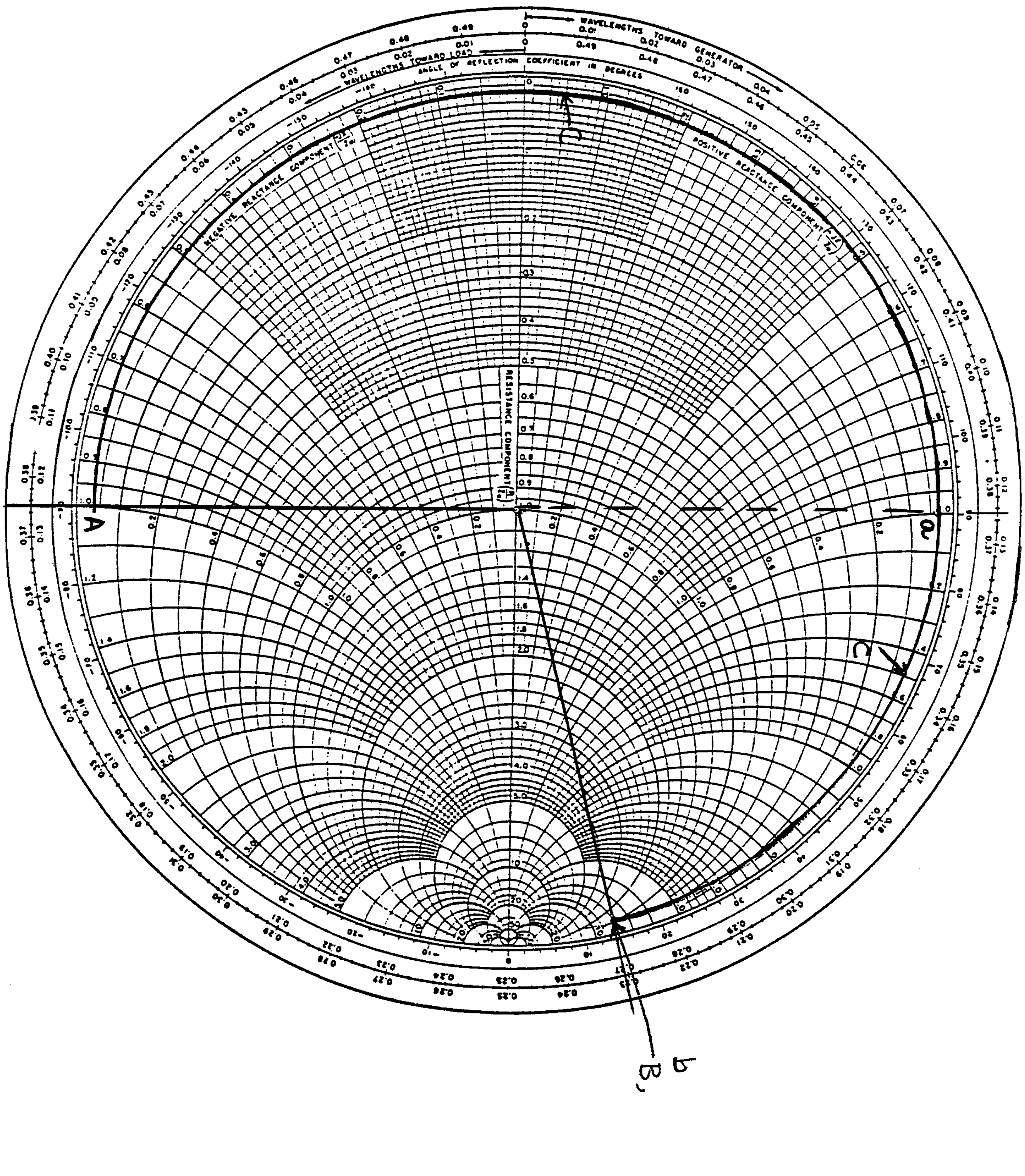
$$jX_{se} = -j\frac{2400}{2} = -j1200$$

This gives the values of series capacitances

$$C_{se} = 4.42 \text{ pF} \tag{6}$$

Example of the Transmission Line Matching Method

f = 30 MHz ; λ_o = 10 m ; Z_a = 10 - j300Ω



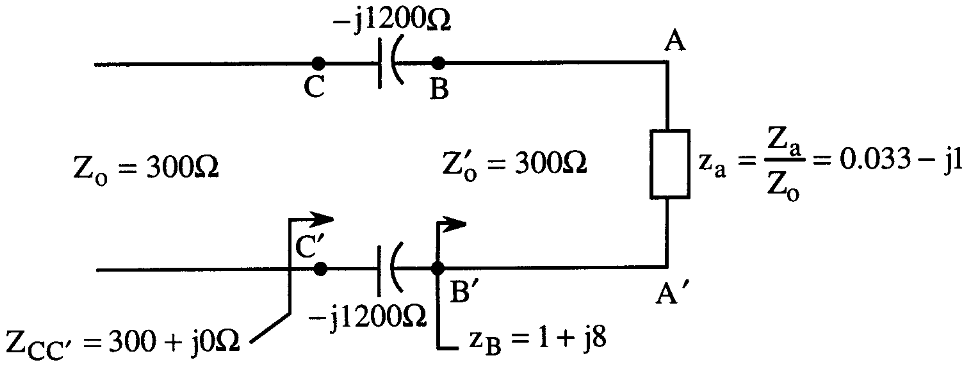


Fig. 2.

Approach B: An alternative design using a shunt element to match the antenna

An undesirable feature of the above design Approach A is that it takes a fairly long length $\ell_{AB} = 0.356\lambda$ over which the transmission line is not matched. For the alternative Approach B, we work in terms of admittances.

$$Y_A = \frac{1}{10 - j300}; \qquad y_A = \frac{1}{0.033 - j1} \cong 0.033 + j1 \tag{7}$$

This is shown by point a on the Smith Chart on page 55.

Now, we need to move only a distance

$$\ell_{ab} = (0.231 - 0.125)\lambda = 0.106\lambda = 1.06 \text{ m} \tag{8}$$

and use (as sketched in Fig. 3) a shunt element to match the line.

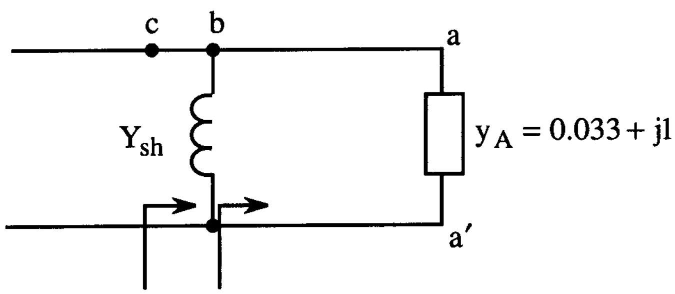


Fig. 3.

$$Y_o \to Y_c = \frac{1}{300} \text{ mho}, \quad y_b = 1 + j8; \quad Y_B = \frac{1}{300} + \frac{j8}{300} \text{ mho}$$

$$-j\,Y_{sh} = -j\frac{8}{300}\text{ mho} \Rightarrow -\frac{j}{\omega\, L_{sh}} \tag{9}$$

$$L_{sh} = \frac{300}{8 \times 2\pi \times 30 \times 10^6} = 0.2\ \mu\text{H} \tag{10}$$

Example 24: A Matching Circuit for an Antenna of a Cellular Telephone

Topology 1

The antenna impedance is given to be 50 - j20Ω. The solid-state source to which this impedance is to be matched has an internal impedance, say 15 + j130Ω. A possible matching circuit is sketched as follows:

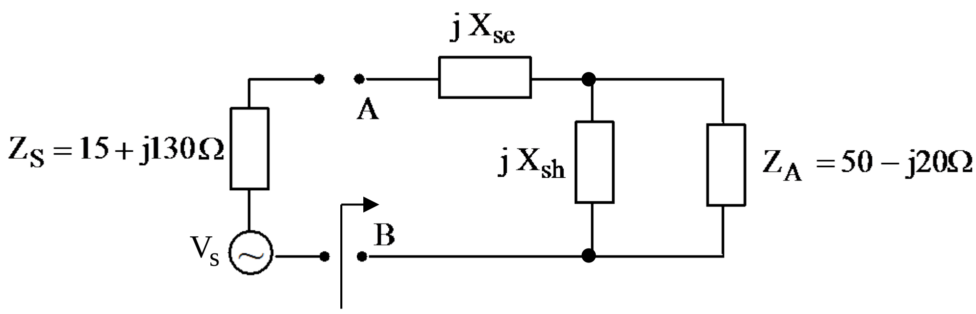


Fig. 1

For maximum power transfer to the antenna

$$Z_{AB} = Z_S^* = 15 - j130\Omega \qquad (1)$$

$$\begin{aligned} Z_{AB} &= \frac{jX_{sh}(50-j20)}{50+j\left(X_{sh}-20\right)} + jX_{se} \\ &= \frac{\left(20\,X_{sh} + j50\,X_{sh}\right)\left[50 - j\left(X_{sh}-20\right)\right]}{(50)^2 + \left(X_{sh}-20\right)^2} + j\,X_{se} \\ &= 15 - j130\Omega \end{aligned} \qquad (2)$$

Equating real parts on both sides of Eq. 2

$$1000\,X_{sh} + 50\,X_{sh}\left(X_{sh}-20\right) = 15\left[2900 + X_{sh}^2 - 40\,X_{sh}\right]$$

$$35\,X_{sh}^2 + 600\,X_{sh} - 43{,}500 = 0 \qquad (3)$$

$$\begin{aligned} X_{sh} &= \frac{-600 \pm \sqrt{(600)^2 + 4\times 35\times 43{,}500}}{70} = \frac{-600 \pm 2540}{70} \\ &= -44.86\,; \quad +27.71\Omega \end{aligned}$$

Taking the capacitive shunt reactance -j44.86Ω and equating the imaginary parts on both sides of Eq. 2, we get

$$X_{se} = -j104.6\Omega = \frac{1}{j\omega C_{se}}$$

$$|X_{sh}| = \frac{1}{\omega C_{sh}} = 44.86\Omega$$

For f = 900 MHz

$$C_{sh} = 3.94 \text{ pF}$$

$$|X_{se}| = \frac{1}{\omega C_{se}} = 104.6\Omega$$

$$C_{se} = 1.69 \text{ pF}$$

The matching circuit for Topology 1 is as follows:

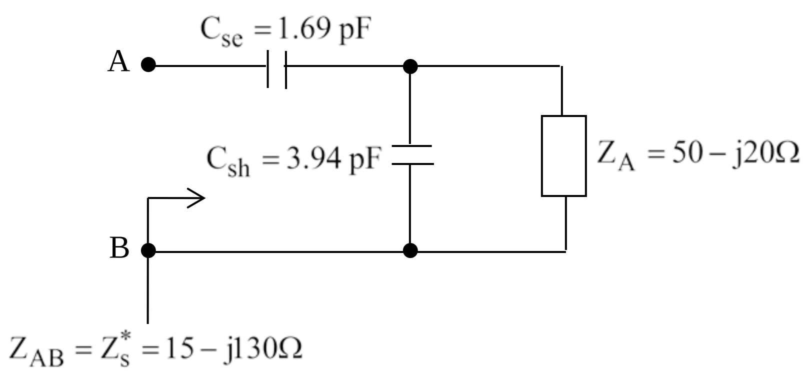


Fig. 2

Topology 2

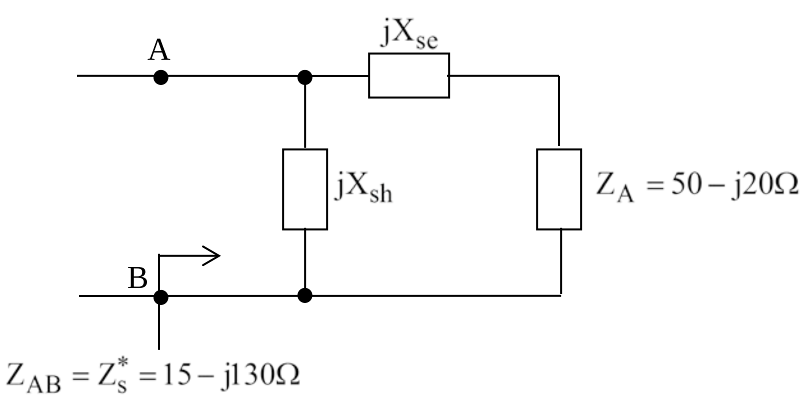


Fig. 3

This problem may be easier to solve in terms of admittances

$$Y_{AB} = \frac{1}{15 - j130} = \frac{15 + j130}{(15)^2 + (130)^2} = \frac{1}{50 + j(X_{se} - 20)} + \frac{1}{jX_{sh}} \qquad (4)$$

Equating real parts

$$\frac{15}{17{,}125} = \frac{50}{(50)^2 + (X_{se} - 20)^2} \qquad (5)$$

$$2500 + (X_{se} - 20)^2 = \frac{50 \times 17125}{15} = 57{,}083$$

$$X_e = 253.6; \qquad -213.6\Omega$$

Taking the sines inductance

$$X_{se} = 253.6\Omega \Rightarrow L_{se} = 44.85 \text{ nH}$$

$$X_{sh} = -85.59\Omega \Rightarrow C_{sh} = 2.06 \text{ pF}$$

Implications for Power Transfer

a. Without conjugate matching, for an oscillator voltage $V_s = 2V$ RMS power

Power transferred to the load = $I_{rms}^2 R_A = \frac{(2)^2}{(15+50)^2 + (130-20)^2} \times 50 = 12.25 \text{ mW}$

b. With conjugate matching

Power transferred to the load = $I'_{rms}{}^2 \operatorname{Re}\left|Z_s^*\right| = \frac{(2)^2}{(15+15)^2} \times 15 = 66.7 \text{ mW}$

Needed for 600 mW power transferred to the load

$$V_s = 6V \text{ RMS}$$

A REACTIVE THREE-ELEMENT CIRCUIT FOR ANTENNA MATCHING

Example 25: A reactive three-element network is a versatile circuit for matching power onto the antenna. To illustrate the procedure, let us look at the circuit of Fig. 1. The antenna equivalent impedance is $R_A + jX_A$.

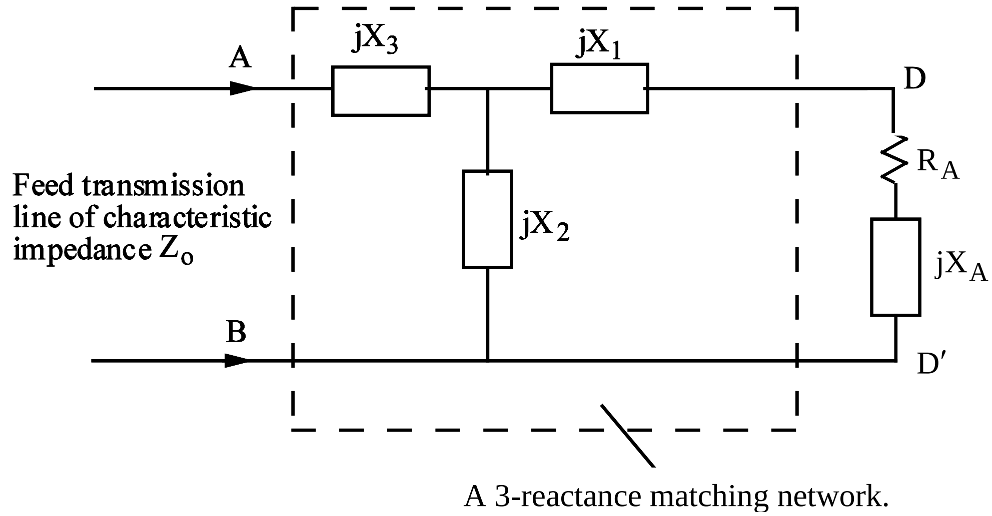


Figure 1

In order to match power into the antenna, it is necessary that the impedance of the network between points A and B be purely resistive and have the same value as Z_o, the characteristic impedance of the transmission line.

From Fig. 1, the expression for the impedance Z_{AB} can be written as:

$$Z_{AB} = \frac{\left(R_A + jX_A + jX_1\right) jX_2}{R_A + jX_A + jX_1 + jX_2} + jX_3 \tag{1}$$

We select X_1 and X_2 such that the reactance in the denominator of the first term is zero, i.e.,

$$X_A + X_1 + X_2 \equiv 0 \tag{2}$$

Equation 1 can then be rewritten as:

$$Z_{AB} = \frac{\left(R_A - jX_2\right) jX_2}{R_A} + jX_3 \tag{3}$$

We select X_2 such that

$$\frac{X_2^2}{R_A} = Z_o \tag{4}$$

and X_3 such that

$$X_3 = -X_2 \tag{5}$$

This would then give

$$Z_{AB} = Z_o + j0$$

and the antenna would then be matched onto the transmission line.

To illustrate the procedure by a *numerical example*, let us say that the antenna is a monopole and its impedance Z_A has been calculated and found to be 1.5 - j460Ω.

Let us take Z_o = 300 ohms (we must, of course, make sure that the diameter of the feeder line is not overly thin for the current-carrying requirement). From Eq. 4,

$$X_2 = \pm\sqrt{1.5 \times 300} = \pm 21.2\Omega \tag{6}$$

The upper sign corresponds to an inductance L = 21.1/ω and the lower sign corresponds to a capacitance

$$C = \frac{1}{\omega \times 21.2} \ .$$

We can use either type.

Case 1: For inductive element X_2

$$jX_2 = j\omega L_2 = j21.2\Omega$$

If ω is prescribed, L_2 can be calculated. From Eq. 2,

$$X_1 = - X_2 - X_A = -21.2 + 460$$

$$= 438.8\Omega$$

This implies an inductor for jX_1. From Eq. 5,

$$X_3 = - 21.2\Omega$$

One possible 3-reactance matching network is, therefore, shown in Fig. 2.

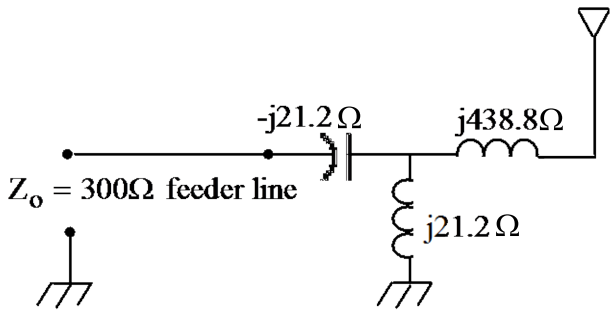


Figure 2

Case 2: For capacitive element X_2

$$jX_2 = \frac{1}{j\omega C_2} = -j21.2\Omega$$

From Eq. 2 on p. 60 of Class Notes,

$$X_1 = -X_2 - X_A = +21.2 + 460$$

$$= 481.2\Omega$$

$$jX_1 = j\omega L_1 = j481.2\Omega \quad \text{(an inductor)}$$

From Eq. 5 on p. 60 of Class Notes,

$$jX_3 = -\,jX_2 = +\,j21.2\Omega \quad \text{(also an inductor)}$$

and a second possible 3-reactance matching network is shown in Fig. 3.

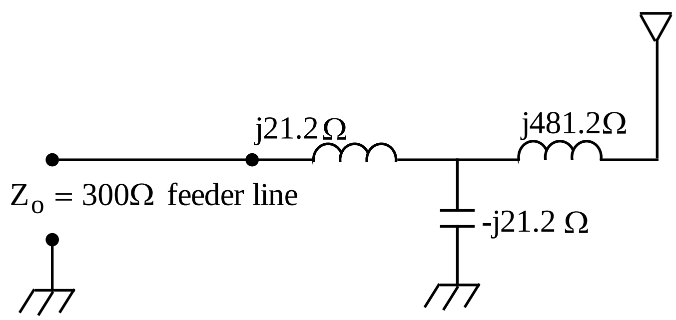
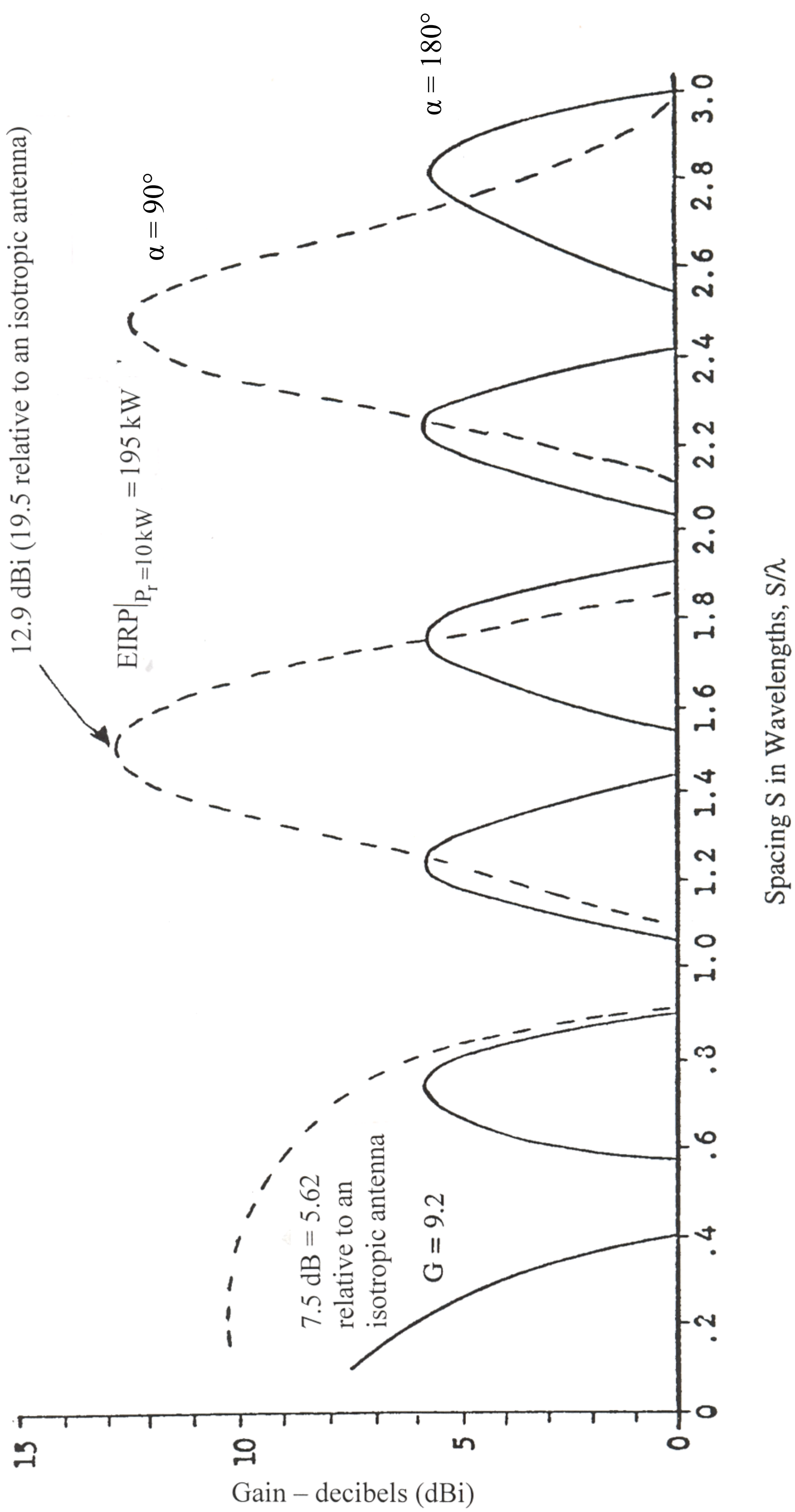


Figure 3

Gain in the axial direction for corner reflector antennas with infinite reflectors



Gain of a dipole antenna placed in a corner reflector of corner angles α = 180° (flat reflector) and α = 90° (90° corner reflector).

H. Jasik, *Antenna Engineering Handbook*, McGraw Hill & Co.

From: K. F. Lee, Principles of Antenna Theory, John Wiley & Sons, 1984

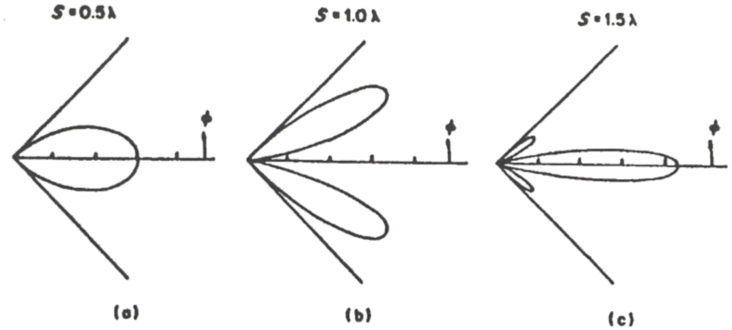


Figure 8.16 Patterns of gain in field intensity in the direction $\theta = 90°$, $\phi = 0$ of square corner reflector antennas with dipole-to-apex spacing of (a) 0.5λ, (b) 1.0λ, and (c) 1.5λ (Source: Kraus (1950), *Antennas*. Reproduced by permission of McGraw-Hill)

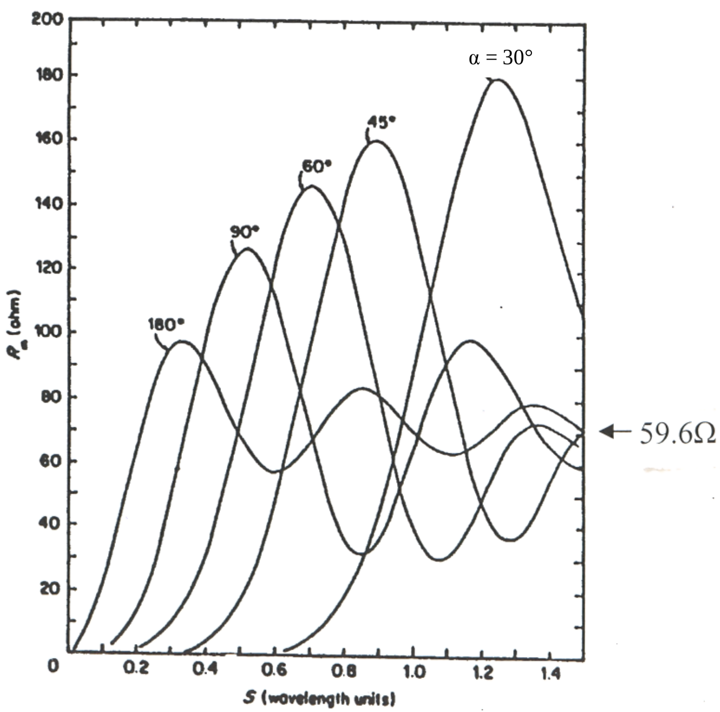


Figure 8.17 Driving-point resistance of corner reflector antenna (Source: Ng and Lee (1982), *IEE Proc.*, 129, Pt H, 11–17. Reproduced by permission of the Institute of Electrical Engineers, England)

Figure 8.15 shows the gain in field intensity versus spacing in the direction $\phi = 0$ for $\beta = 180°$, $90°$, and $60°$. Two curves are shown for each corner angle. The full curve in each case is computed for $R_L = 0$ while the broken curve is for an assumed loss resistance $R_L = 1$ ohm.

The pattern in the horizontal plane for a square ($\beta = 90°$) corner reflector is shown in Figure 8.16 for three values of the dipole-to-corner spacing S. For $S = 0.5\lambda$, the pattern is single-lobed. A two-lobed pattern results if S is increased to 1.0λ. If S is further increased to 1.5λ, the pattern has a major lobe in the $\phi = 0$ direction but with minor lobes present.

The input or driving-point impedance of the antenna is given by

$$Z_{in} = R_{in} + jX_{in} = V(0)/I(0).$$

where $V(0)$ and $I(0)$ are the voltage and current at the terminals of the driven dipole. Since the currents in the elements are equal in magnitude, we have

$$Z_{in} = Z_{00} + \sum_{i=1}^{2N-1} Z_{0i}(-1)^i \qquad (8.61)$$

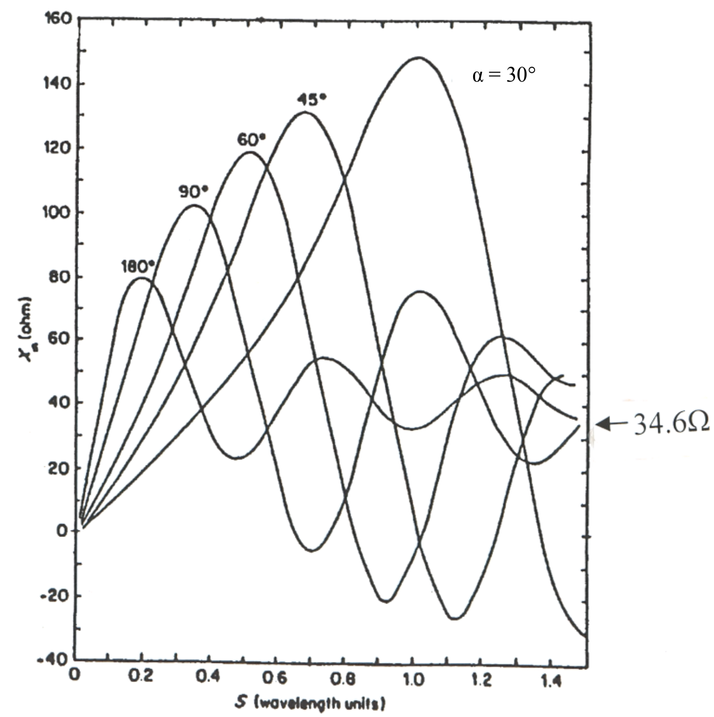
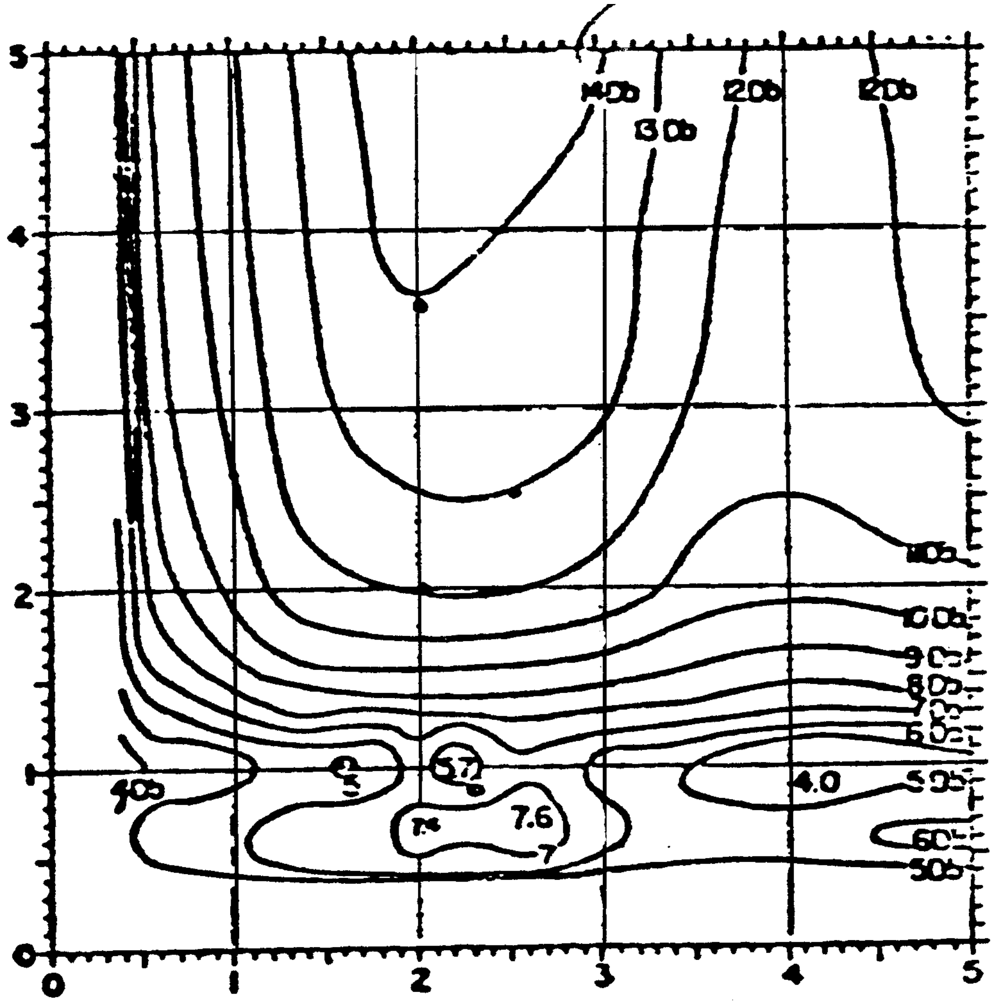


Figure 8.18 Driving-point reactance of corner reflector antenna (Source: Ng and Lee (1982), *IEE Proc.*, 129, Pt H, 11–17. Reproduced by permission of the Institute of Electrical Engineers, England)

H. V. Cottany and A. C. Wilson, "Gains of Finite Size Corner Reflector Antennas," *IEEE Transactions on Antennas and Propagation,* Vol. AP-6, 1958, pp. 366-369.

Contours of constant gain for a 90º corner reflector

25.12 × 1.64 = 41.2



Length of the reflecting planes in wavelengths, L/λ

Width of the reflecting planes in wavelengths, W/λ

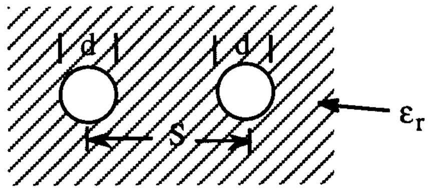
Maximum for infinite reflectors = 12.9 dB (19.5)

Some Commonly Used Feeder Lines for Antennas

1. Twin Wire Transmission line

$$Z_o = \frac{120}{\sqrt{\varepsilon_r}} \ell n \left(\frac{2S}{d} \right)$$

(Replace ε_r by ε_{eff} for relatively thin dielectric sheathing)

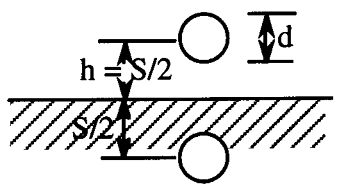


Example:

$$\frac{2S}{d} = 12.2 \quad \text{for } Z_o = 300\Omega \text{ (air-filled line)}$$

2. Wire Above Ground Transmission Line

$$Z_o = \frac{60}{\sqrt{\varepsilon_r}} \ell n \left(\frac{2S}{d} \right) = \frac{60}{\sqrt{\varepsilon_r}} \ell n \left(\frac{4h}{d} \right)$$

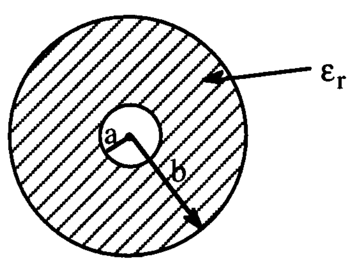


Example:

$$\frac{h}{d} = 3.05 \quad \text{for } Z_o = 150\Omega \text{ (air-filled line)}$$

3. Coaxial Line

$$Z_o = \frac{60}{\sqrt{\varepsilon_r}} \ell n \left(\frac{b}{a} \right)$$



Example:

$$\frac{b}{a} = 3.345 \quad \text{for} \quad \varepsilon_r = 2.1 \text{ (Teflon) coaxial line of } Z_o = 50\Omega$$

Some of the other transmission lines useful for printed antennas are:

a. Miscrostripline

b. Slot line etc.

Ground Effect on Radiation Pattern of an Antenna

We have previously considered the effect of ground for the radiation from a vertical monopole antenna. The net effect was that the monopole antenna of length L/2 radiates electromagnetic fields much like a dipole of length L albeit for the upper half plane i.e. for field points above ground.

For a horizontal dipole antenna placed at a distance h from the ground as sketched in Fig. 1, an image antenna 1′ is created, which has a current excitation that is equal in magnitude (for high conductivity ground) but 180º out of phase with that in the installed antenna #1.

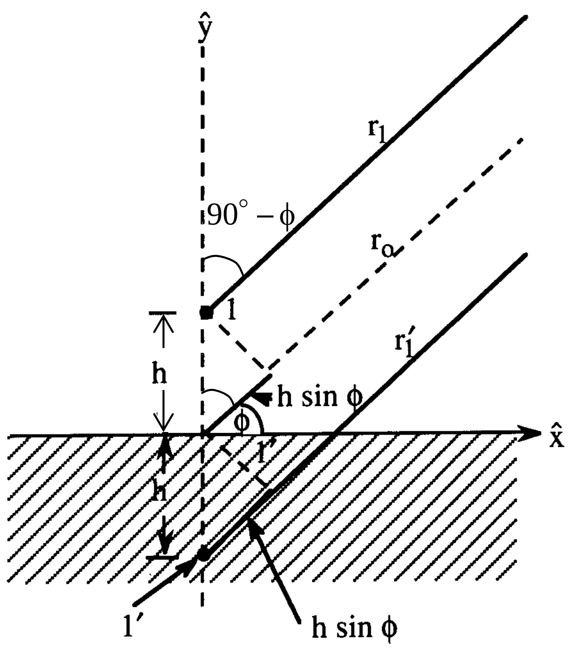


Fig. 1. A horizontal dipole antenna above ground.

From Eq. 10 on p. 24 of the Class Notes, this can be considered as a two-element array ($N_y = 2$) with a phase difference $\alpha_y = \pi$ or 180º.

$$\vec{E}_T = \vec{E}_1 |AF|_y = \vec{E}_1 \frac{\sin\left(2\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} = \vec{E}_1 2\cos\left\{\frac{1}{2}\left(2\beta h \sin\theta \sin\phi + \pi\right)\right\}$$
$$= \vec{E}_1 2\sin\left(\beta h \sin\theta \sin\phi\right) \tag{1}$$

neglecting the phase factors both in writing $|AF|_y$ and $\vec{E}_T$. Note that Eq. 1 could also have been written by following a procedure similar to that for Eq. 4 on page 24 of the Class Notes.

$$\vec{E}_T = \vec{E}_1 + \vec{E}_i = E_1\left[1 - e^{-j\beta(r_1' - r_1)}\right] = E_1\left[1 - e^{-j\beta(2h\sin\theta\sin\phi)}\right]$$
$$= \vec{E}_1\left[e^{-j\beta h \sin\theta\sin\phi} - e^{-j\beta h\sin\theta\sin\phi}\right] = 2\,E_1 \sin\left(\beta h \sin\theta \sin\phi\right) \tag{2}$$

ignoring the phase factors, as also done in writing Eq. 1. From Eqs. 1 and 2

$$|AF| = 2\sin\left(\beta h \sin\theta\sin\phi\right) \Rightarrow 2\sin\left(\beta h \sin\phi\right) \tag{3}$$

for $\theta = \pi/2$ i.e. xy plane.

For <u>maxima of radiation</u>

$$\beta h \sin\phi_o = \pm\frac{\pi}{2}, \quad \pm 3\frac{\pi}{2}, \quad \square \tag{4}$$

For <u>first nulls of radiation</u>

$$\beta h \sin\phi_{FN} = 0, \quad \pm\pi, \quad \pm 2\pi, \quad \square \tag{5}$$

Example 26

a. Calculate the spacing h to ground for a half-wave dipole antenna if the maximum of radiation is desired for angle $\phi_o = 30^o$ off the horizon.

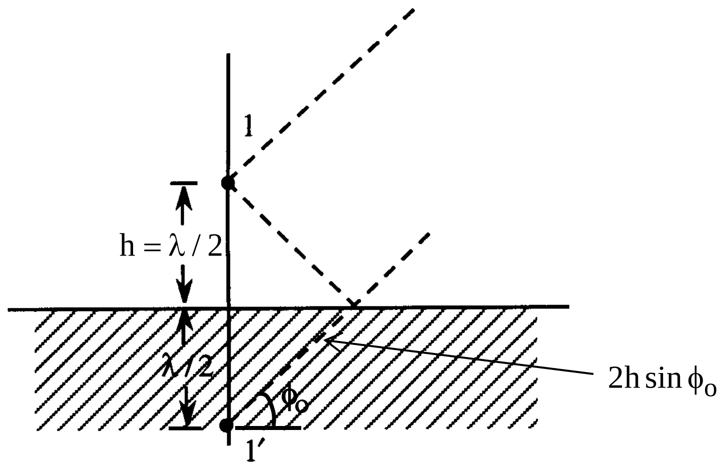
b. Calculate the directions of maximum and zero radiation for the selected h.

c. Calculate the gain of the antenna, without and with mutual impedance effects.

<u>Solution:</u> From Eq. 4 for $\phi_o = 30^o$, $\sin\phi_o = 0.5$

a.

$$h = \frac{\lambda}{4\sin\phi_o}, \quad \frac{3\lambda}{4\sin\phi_o}, \quad \square$$
$$= \frac{\lambda}{2}, \quad \frac{3\lambda}{2}, \quad \frac{5\lambda}{2}, \quad \square \tag{6}$$



In order to keep the number of principal maxima to a minimum number, we select the smallest spacing to the ground plane i.e.

$$h = \lambda/2 \tag{7}$$

b. For this spacing itself, we note from Eq. 4 that the directions of maximum radiation are:

$$\beta h \sin \phi_o = \pi \sin \phi_o = + \frac{\pi}{2} \tag{8}$$

$$\phi_o = 30^\circ \text{ (wanted)}, \quad \phi_o = 150^\circ \text{ (unwanted)}$$

Negative sign is ignored in Eq. 8 since that gives angles ϕ_o = -30º, -150º (both into the ground).

We will see later how to eliminate the unwanted radiation for ϕ_o = 150º. If we had taken a larger h of say 3 λ/2 from Eq. 6, we would have had many more directions of maximum radiation.

For directions of first null, from Eq. 5, ϕ_{FN} = 0 and $\sin^{-1}(1)$ or 0 and 90º for the principal maximum of ϕ_o = 30º, and ϕ_{FN} = 180º and $\sin^{-1}(1)$ or 180º and 90º for the principal maximum at ϕ_o = 150º. The radiation pattern is sketched in Fig. 2.

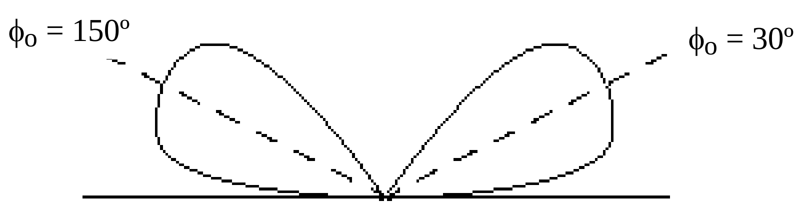


Fig. 2.

c. Ignoring mutual impedance effects $R_{a1} = 73\Omega$ (same as for an isolated half wave dipole). From Eq. 1

$$E_{max} = 2E_1; \quad S_{max} = \frac{E_{max}^2}{2\eta} = 4S_1$$

$$\text{Gain} = 4G_1 = 4 \times 1.64 = 6.56$$

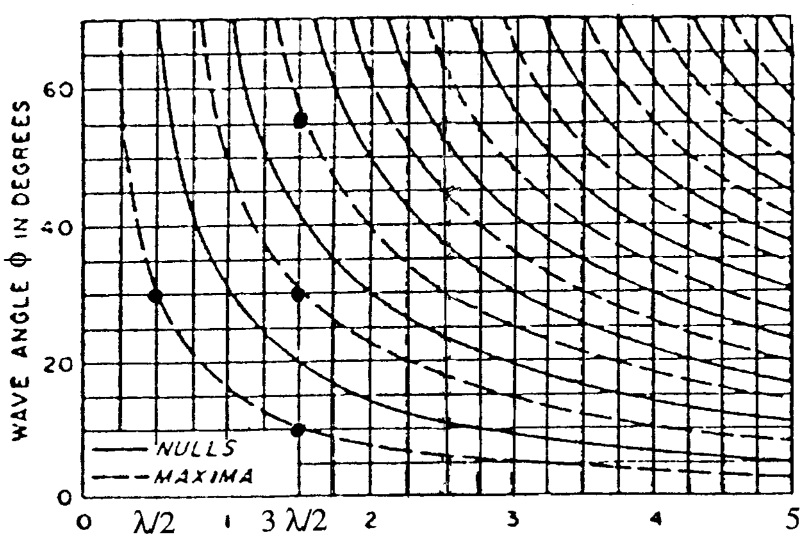


Fig. 3. Angles ϕ of maximum and zero radiation for a horizontal dipole antenna above ground (From Eq. 1, 2, or 3).

Elimination of Unwanted Principal Lobes of Radiation

As seen in Example 24, there is an unwanted principal lobe of radiation for $\phi_o = 150^o$ that we would like to eliminate leaving thereby one and only one principal lobe of radiation for the desired direction $\phi_o = 30^o$. A possible solution for this problem is as sketched in Fig. 4.

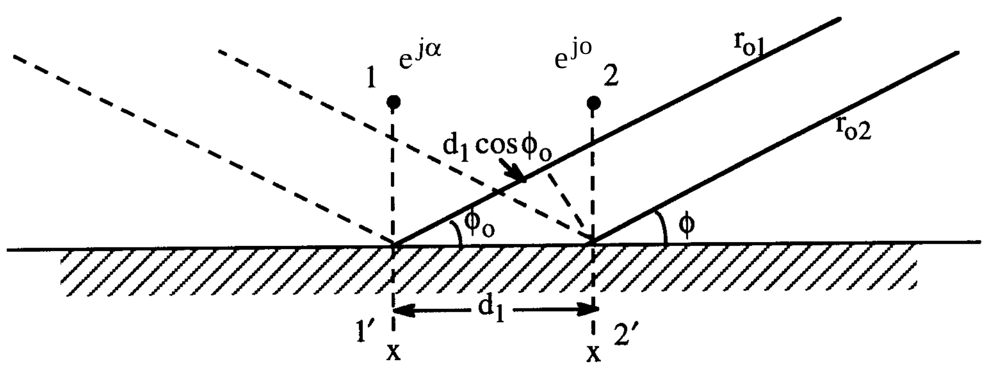


Fig. 4. An arrangement of two horizontal dipoles above ground.

We take two horizontal dipoles 1, 2 above ground. Distance to ground h is the same for both dipoles 1 and 2. Shown in Fig. 4 also are the two image antennas $1', 2'$. Assuming that antenna #1 is leading in phase by α (i.e. antenna 2 is lagging in phase by α).

$$\alpha - \beta\, d_1 \cos \phi_o = 0 \tag{9}$$

for addition of signals along $\phi_o = 30^\circ$ principal lobe.

$$\alpha + \beta\, d_1 \cos \phi_o = \pi \tag{10}$$

for complete cancellation of radiation in the back direction.

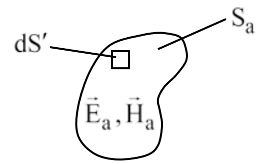
Note in both Eqs. 9 and 10, $\phi_o = 30^\circ$ and $d_1 \cos \phi_o = 0.866\, d_1$. From Eqs. 9 and 10, both the unknown α and d_1 can now be found:

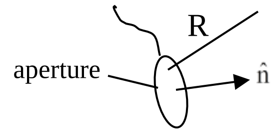
$$\alpha = \frac{\pi}{2} = 90^\circ \text{ (phase lead angle for antenna \#1)} \tag{11}$$

$$\beta d_1 \cos \phi_o = \frac{\pi}{2} \quad \Rightarrow \quad d_1 = \frac{\lambda}{4 \cos \phi_o} = 0.289\lambda \tag{12}$$

This arrangement would cancel the principal lobe for $\phi_o = 150^\circ$ (in Fig. 2) while reinforcing the principal lobe for the $\phi_o = 30^\circ$ angle of radiation.

p. 349 Text **General Theory of Aperture Antennas (or Displacement Current Antennas).** For comparison, see also the General Theory of Conduction Current Antennas on p. 44 of the Text or p. 2 of Class Notes.





Source of Fields $\vec{H}_a$ | Define Equivalent surface currents | Source of Fields $\vec{E}_a$

$$\vec{J}_s = \hat{n} \times \vec{H}_a \qquad (9\text{-}10)$$

$$\vec{A} = \frac{\mu}{4\pi} \int_{S_a} \frac{\vec{J}(\vec{r}')}{R} e^{j(\omega t - \beta R)} dS' \qquad R = \vec{r} - \vec{r}'$$

$$= \frac{\mu\, e^{-j\beta r}}{4\pi r}\, \hat{n} \times \underbrace{\boxed{\int_{S_a} \vec{H}_a e^{j\beta \hat{r}\cdot\vec{r}'} dS'}}_{Q} \qquad (9\text{-}12)$$

$$\vec{H}_1 = \frac{\nabla \times \vec{A}}{\mu} = -\frac{j\vec{\beta} \times \vec{A}}{\mu} \qquad (1)$$

$$\vec{E}_1 = \frac{\nabla \times \vec{H}_1}{j\omega\varepsilon_o} = -\frac{j\vec{\beta} \times \vec{H}_1}{j\omega\varepsilon_o} = -j\omega\vec{A} \qquad (2)$$

$$\vec{M}_s = \vec{E}_a \times \hat{n} \qquad (9\text{-}11)$$

$$\vec{F} = -\frac{\varepsilon}{4\pi} \int_{S_a} \frac{\vec{M}_s(\vec{r}')}{R} e^{j(\omega t - \beta R)} dS'$$

$$= \frac{\varepsilon\, e^{-j\beta r}}{4\pi r}\, \hat{n} \times \underbrace{\boxed{\int_{S_a} \vec{E}_a e^{j\beta \hat{r}\cdot\vec{r}'} dS'}}_{P} \qquad (9\text{-}13)$$

$$\vec{E}_2 = -\frac{\nabla \times \vec{F}}{\varepsilon} = \frac{j\beta\hat{r} \times \vec{F}}{\varepsilon} = -\frac{j\beta\vec{F} \times \hat{r}}{\varepsilon} = -j\omega\eta\, \vec{F} \times \hat{r} \qquad (3)$$

$$\vec{E}_T = \vec{E}_1 + \vec{E}_2 = -j\omega\vec{A} - j\omega\eta\ \vec{F} \times \hat{r} \qquad (9\text{-}16)$$

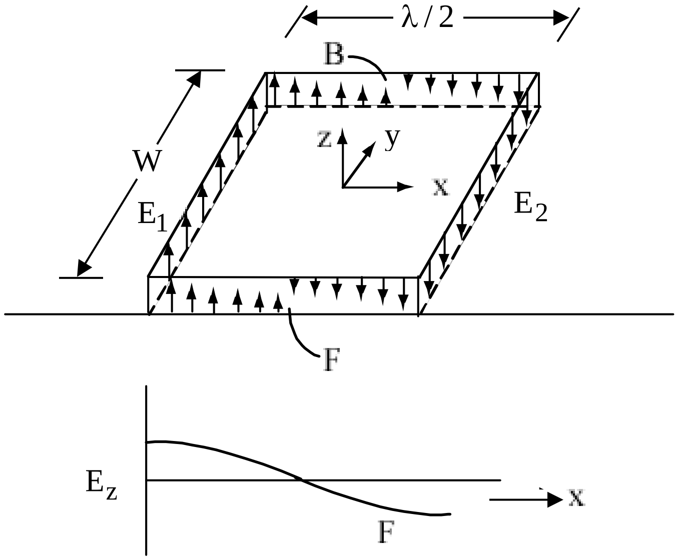
$$\vec{E}_T = \vec{E}_1 + \vec{E}_2 = -\frac{j\beta\, e^{-j\beta r}}{4\pi r}\, \hat{r} \times \int_{S_a} \left[\hat{n} \times \vec{E}_a - \eta\, \hat{r} \times \left(\hat{n} \times \vec{H}_a \right) \right] e^{-j\beta \hat{r}\cdot\vec{r}'} dS' \qquad (9\text{-}17)$$

$$\vec{H}_T = \frac{\nabla \times \vec{E}_T}{-j\omega\mu_o} = -\frac{j\beta\, \hat{r} \times \vec{E}_T}{j\omega\mu_o} = -\frac{\hat{r} \times \vec{E}_T}{\eta}$$

$$\vec{S} = \frac{1}{2} \mathrm{Re}\left(\vec{E} \times \vec{H}^* \right) = \frac{\vec{E}_T \cdot \vec{E}_T^*}{2\eta}\, \hat{r}$$

$$\text{Total radiated power} = \int_{\text{sphere}} \vec{S} \cdot d\vec{S}$$

p. 466 Text **A Rectangular Microstrip Patch Antenna**



For the front edge F; also for the back edge B

Notes

1. Note that the E-fields have no variation in the y direction; hence are identical for the front and back edges separated by width W.

2. Because of a separation distance of λ/2, the E-fields at edges E_1 and E_2 are 180º out of phase; also no variation in y-direction.

Fig. 1. Distributions and variations of electric fields at the four edges of the patch antenna.

$$\vec{M}_s = \vec{E}(s) \times \hat{n} \tag{9-2b}$$

p. 346 Text

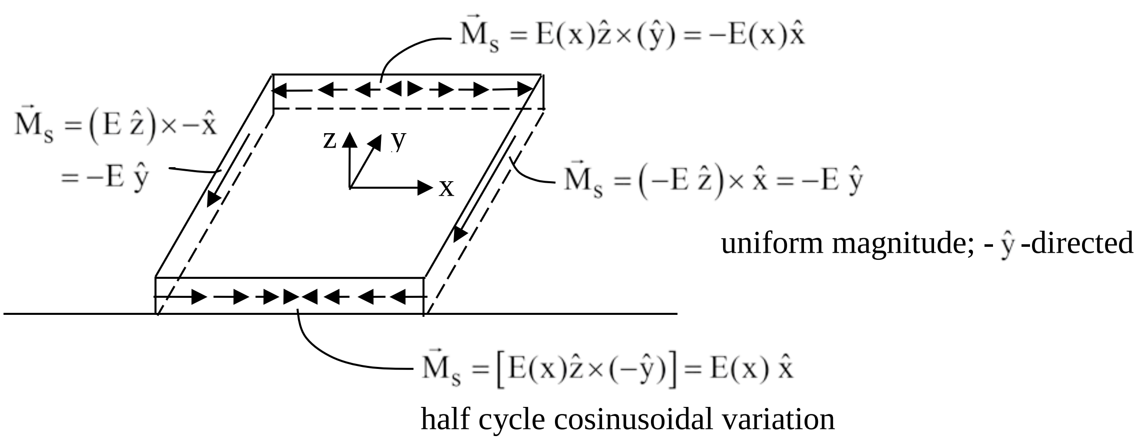


Fig. 2. Equivalent surface currents at the four edges.

Maximum radiation in z-direction (normal to the patch) due to two **uniform** magnitude "dipoles" corresponding to edges E_1 and E_2; radiation from edges F and B i.e. the front and back edges of Fig. 1 cancels out.

The array factor for the equivalent current dipoles at edges E_1, E_2 can be written from Eqs. 8 and 9 on page 25 of the Class Notes. For an x-directed array of two elements $N = 2$; $\alpha_x = 0$;

$$\psi = \beta L \sin\theta \cos\phi + \cancel{\alpha_x}^{\,0}$$

$$\text{Normalized AF} = \frac{\sin\left(\frac{N\psi}{2}\right)}{2\sin\left(\frac{\psi}{2}\right)} = \frac{2\sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)}{2\sin\frac{\psi}{2}} = \cos\left(\frac{\beta L}{2}\sin\theta\cos\phi\right) \qquad (1)$$

For a two-element (two-edge "currents") antenna array

$$\vec{E}_T = \vec{E}_o \text{ AF}$$

For a **uniformly-excited** rectangular aperture of length W (e.g. edges E_1, E_2), from Eqs. 9-36a, b (note that equivalent current $\vec{M}_s \parallel -\hat{y}$ here rather than parallel to $\hat{x}$ on page 354 of the text)

$$E_\theta = E_o \cos\phi \, f(\theta, \phi) \qquad (11\text{-}5a)$$

$$E_\phi = -E_o \cos\theta \sin\phi \, f(\theta, \phi) \qquad (11\text{-}5b)$$

where

$$f(\theta,\phi) = \frac{\sin\left(\frac{\beta W}{2}\sin\theta\sin\phi\right)}{\frac{\beta W}{2}\sin\theta\sin\phi}\text{AF}$$

($W \leftarrow L_y$)

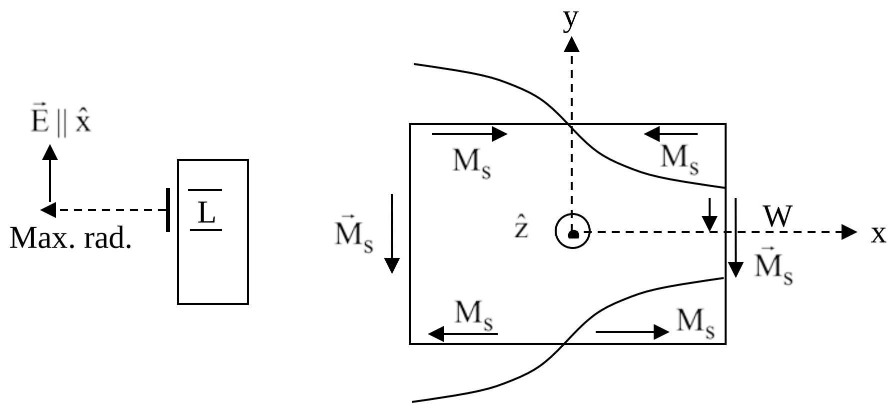
$$= \frac{\sin\left(\frac{\beta W}{2}\sin\theta\sin\phi\right)}{\frac{\beta W}{2}\sin\theta\sin\phi}\cos\left(\frac{\beta L}{2}\sin\theta\cos\phi)\right) \qquad (11\text{-}5c)$$

In Eqs. 9-36a, b

$$\frac{\sin\left(\frac{\beta L_z}{2}u\right)}{\frac{\beta L_z}{2}u} \to 1 \qquad \text{since} \quad t \ll \lambda$$

($L_z \leftarrow$ thickness t)

p. 470 Text **Microstrip Patch Antenna**



For x-z plane ($\phi = 0^\circ$) or **E-plane**

$\vec{E}$ field is $\hat{\theta}$-directed with components in x- and z-directions

$$E_\theta = E_o f(\theta, \phi); \qquad E_\phi = 0$$

For direction of maximum radiation, $\vec{E} \parallel \hat{x}$

$$F_E(\theta) = \cos\left(\frac{\beta L}{2}\sin\theta\right) \tag{11-6a}$$

Maximum for $\theta = 0$ i.e. along z-direction.

BWFN:

$$\frac{\beta L}{2}\sin\theta = \frac{\pi}{2} \quad \rightarrow \quad \theta_{FN} = \sin^{-1}\left(\frac{\lambda}{2L}\right)$$

For $L \simeq \frac{\lambda}{2}$, for xz or **E-plane**

$$\text{BWFN} = 2\sin^{-1}\left(\frac{\lambda}{2L}\right)$$
$$= 180^o$$

HPBW:

$$\frac{\beta L}{2}\sin\theta = \frac{\pi}{4} \quad \rightarrow \quad \theta_{HP} = \sin^{-1}\left(\frac{\lambda}{4L}\right)$$

$$\text{HPBW} = 2\sin^{-1}\left(\frac{\lambda}{4L}\right) \rightarrow 2\sin^{-1}\left(\frac{1}{2}\right) \rightarrow 60^\circ$$

For y-z plane ($\phi = 90^\circ$)

$$F_H(\theta) = \cos\theta \frac{\sin\left[\frac{\beta W}{2}\sin\theta\right]}{\frac{\beta W}{2}\sin\theta} \qquad (11\text{-}6b)$$

Maximum for $\theta = 0$

$$\vec{E} = -E_o \blacktriangle \underbrace{\cos\theta\, F(\theta,\phi)}_{F_H(\theta)} \blacktriangle$$

For yz or H-plane

$$\frac{\beta W}{2}\sin\theta_{FN} = \pi; \qquad \theta_{FN} = \sin^{-1}\left(\frac{\lambda}{W}\right)$$

$$BWFN = 2\sin^{-1}\left(\frac{\lambda}{W}\right)$$

$$Z_A = 90\frac{\varepsilon_r^2}{\varepsilon_r - 1}\left(\frac{L}{W}\right)^2 \qquad (11\text{-}7)$$

for a half-wave rectangular patch antenna.

For Duroid ($\varepsilon_r = 2.2$) and $\frac{W}{L} = 2.7$

$$Z_A = 50\Omega$$

$$\theta_{FN}\big|_{yz\ plane} = \sin^{-1}\left(\frac{1}{1.35}\right) = 47.8^\circ$$

$$BWFN = 2\ \sin^{-1}\left(\frac{\lambda}{W}\right) = 95.6^\circ$$

Table 7.1. Radiation characteristics of commonly-used horn antennas.

Type of Horn	Property that is Optimized for a Given Length	Optimum Properties	Half-power Beam Widths in Degrees		Directive Gain
			H (or xz) Plane	E (or yz) Plane	
Pyramidal	Gain	$A = \sqrt{3L\lambda}$ $B = 0.81A$ Gain = 15.3 L/λ (optimum)	$\frac{80}{(A/\lambda)}$	$\frac{53}{(B/\lambda)}$	$0.51\frac{4\pi AB}{\lambda^2}$ **(9-96)**
Sectoral H-plane horn	Beam width in H-plane	$A = \sqrt{3L\lambda}$	$\frac{78}{(A/\lambda)}$ **(9-124)**	$\frac{51}{(B/\lambda)}$	$0.63\frac{4\pi AB}{\lambda^2}$
Sectoral E-plane horn	Beam width in E-plane	$B = \sqrt{2L\lambda}$	$\frac{68}{(A/\lambda)}$	$\frac{54}{(B/\lambda)}$ **(9-138)**	$0.65\frac{4\pi AB}{\lambda^2}$
Conical	Gain	$D = \sqrt{2.8L\lambda}$	$\frac{70}{(D/\lambda)}$	$\frac{60}{(D/\lambda)}$	$0.52\frac{4\pi(\text{area})}{\lambda^2}$

Notation: A is the horn dimension in x direction
B is the horn dimension in y direction
D is the horn diameter
L is the length of the horn from the throat to the aperture

Table 7.2*. Comparative characteristics of parabolic reflectors with different illuminations. (See also Table 9-2, p. 389 Text).

Illumination	3 dB Beam Width in Degrees	Peak Side Lobe Level (dB)	Relative Gain	First Null Position in Degrees
A. Rectangular Aperture of Length L				
$G(x) = \cos^m\left(\frac{\pi x}{L}\right) \text{ for } \lvert x \rvert < \frac{L}{2}$				
m = 0 (uniform)	$50.8^\circ \lambda / L$	-13.2	1.00	$57.3^\circ \lambda / L$
m = 1	$68.8^\circ \lambda / L$	-23	0.81	$85.9^\circ \lambda / L$
m = 2	$83.1^\circ \lambda / L$	-32	0.667	$114.6^\circ \lambda / L$
m = 3	$95.1^\circ \lambda / L$	-40	0.575	$143.2^\circ \lambda / L$
m = 4	$111.2^\circ \lambda / L$	-48	0.515	$171.9^\circ \lambda / L$
B. Circular Aperture of Diameter D				
$G(\rho) = \left\{1 - \left(\frac{2\rho}{D}\right)^2\right\}^m$				
m = 0 (uniform)	$58.4^\circ \lambda / D$	-17.6	1.00	$69.9^\circ \lambda / D$
m = 1	$72.8^\circ \lambda / D$	-24.6	0.75	$92.2^\circ \lambda / D$
m = 2	$84.2^\circ \lambda / D$	-30.7	0.55	$116.3^\circ \lambda / D$
m = 3	$94.5^\circ \lambda / D$	-36.1	0.45	$138.7^\circ \lambda / D$

* M. I. Skolnik, *Radar Handbook*, Chapter 9, McGraw-Hill Book Company, Inc., New York, 1970.