# A more rigorous analysis of spy game (Optional Reading)

**Question 1:** Lets assume someone just rolled a 6 in public and announced *HIT*. Based on this information what is the conditional probability that his color is *blue*?

**Solution:** Before we dive into computing conditional probabilities, lets first make some important observations:

- There are 4 variables in the game: *player color, public dice, private dice* and *result*. *Player color* is one of the following colors: $\{yellow, orange, green, blue, red, purple\}$. *Public dice* possible outcomes are $\{1, 2, 3, 4, 5, 6\}$. *Private dice* possible outcomes are $\{1, 2, 3, 4, 5, 6\}$. The *result* can be one of $\{HIT, MISS\}$.

- If we don't specify the result of a turn, the *player color*, the *public dice* they throw and the *private dice* they throw are all independent. In other words, a player's color can not influence what values they will roll on the dice. The public dice can't influence the private dice and *vice versa*.

  For each of these three variables individually, all outcomes are equally likely and since there are 6 possible outcomes, the probability of each outcome is $1/6$. For instance, $P(player\ color = red) = 1/6$, $P(public\ dice = 4) = 1/6$ and $P(private\ dice = 5) = 1/6$. Then using independence, we can compute:

  $$P\left((player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5)\right) =$$
  $$P\left((player\ color = red)\right) \times P\left((public\ dice = 4)\right) \times P\left((private\ dice = 5)\right) =$$
  $$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6^3}$$

  In general:

  $$P\left((player\ color = x) \cap (public\ dice = y) \cap (private\ dice = z)\right) = \frac{1}{6^3}$$

  as long as $0 \leq y \leq 6$, $0 \leq z \leq 6$ and $x$ is one of the player colors used in the game.

- The *result* variable is a deterministic function of *player color, public dice* and *private dice*. In other words, as soon as we specify values for these three variables, there is no uncertainty in the *result* variable. Another way to think about it: you know your own color so as soon as you roll your public and private dice, the game board dictates what you must announce (*HIT* or *MISS*). Of course, the rules of the game assumes the players are being truthful. Therefore

  $$P\left(result = HIT \mid (player\ color = x) \cap (public\ dice = y) \cap (private\ dice = z)\right) = 0\ or\ 1$$

  as dictated by the layout of colors on the board.

- To compute conditional probabilities, we will have to count the number of outcomes in various events. An intuitive way to do this is to plot the combined sample space of the 4 variables. Drawing a 4-dimensional graph is tricky, but we can do it as shown in the following page.

plot of P{color, public die, private die, result}



● = $\frac{1}{6^3}$

plot of P{color, public die, private die, result}
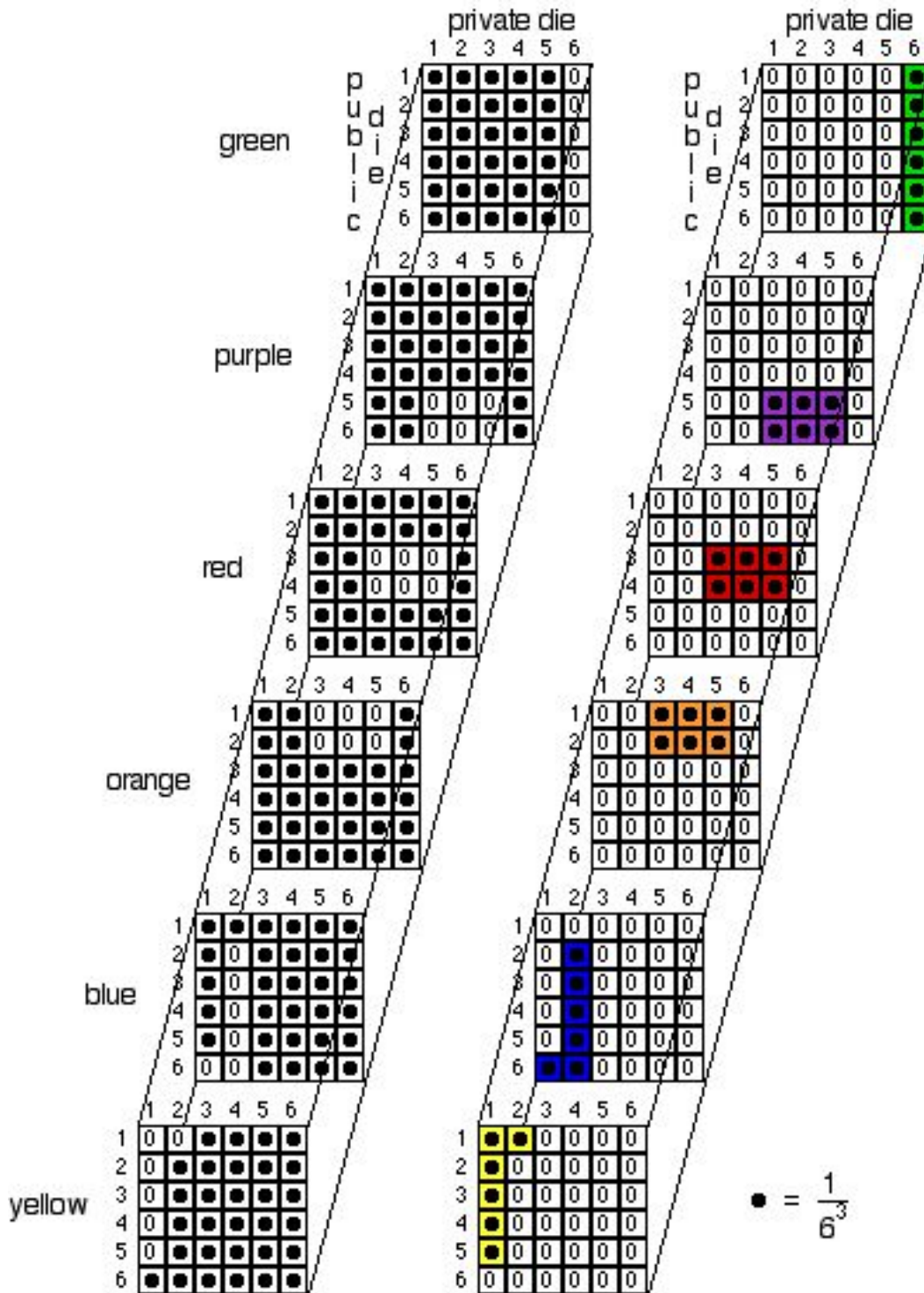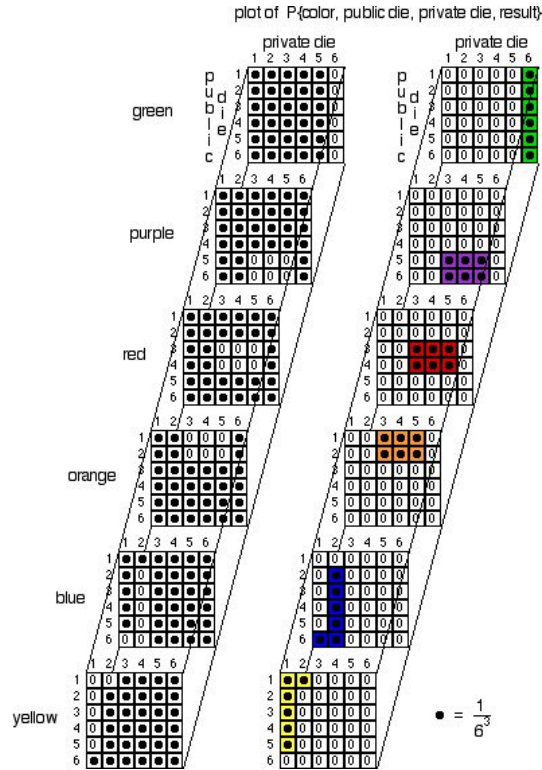
- The plots on the left are for *result=MISS*. The plots on the right are for *result=HIT*.

- The $6 \times 6$ faces from top to bottom correspond to the various player colors.

- Remember there is no uncertainty left in the result once the other 3 variables are known. Still, we treat the result as a separate parameter for making the above plot because this will make counting easier.

- A filled circle indicates a probability of $1/6^3$. A 0 indicates a probability of 0. Example:

$$
\begin{aligned}
P\left((player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5) \cap (result = HIT)\right) &= \\
P\left(result = HIT \mid (player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5)\right) &\times \\
P\left((player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5)\right) &= \\
1 \times \frac{1}{6^3} = \frac{1}{6^3}
\end{aligned}
$$

$$
\begin{aligned}
P\left((player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5) \cap (result = MISS)\right) &= \\
P\left(result = MISS \mid (player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5)\right) &\times \\
P\left((player\ color = red) \cap (public\ dice = 4) \cap (private\ dice = 5)\right) &= \\
0 \times \frac{1}{6^3} = 0
\end{aligned}
$$

- The sum of probabilities of all the squares in the plot must be 1. There are $2 \times 6^3$ squares in the plot. Half of these have probability $1/6^3$ while the other have probability 0. Therefore, the sum of all the probabilities is 1.

We are finally in a position to answer the original question: Someone just rolled a 6 in public and announced *HIT*. Based on this information we want to evaluate the conditional probabilities for him having the various colors. Lets start by considering the color blue. We have to compute:

$$P\left((player\ color = blue) \mid (public\ dice = 6) \cap (result = HIT)\right)$$

From the definition of conditional probability, this is equivalent to:

$$\frac{P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = HIT)\right)}{P\left((public\ dice = 6) \cap (result = HIT)\right)} \tag{1}$$

Lets evaluate the numerator. Notice that the *private dice* outcomes form a partition of the whole sample space. Therefore, using the law of total probability:

$$P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = HIT)\right) =$$

$$\sum_{z=1}^{6} P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = HIT) \cap (private\ dice = z)\right) \tag{2}$$

So, which squares of our plot are involved in this summation? First, the result we are considering is a *HIT* so we are looking at the plots on the right side. Second, the color we are considering is **blue** so we are now looking at the second from bottom $6 \times 6$ face on the right side. Third, the **public dice** is 6 so we are looking at the bottom row of that face. The summation $z = 1$ to $z = 6$ for the **private dice** takes us across all the squares in this row. Notice that in this row there are two filled circles each corresponding to a probability of $1/6^3$ and four 0's. Therefore, equation (2) resolves as

$$\frac{1}{6^3} + \frac{1}{6^3} + 0 + 0 + 0 + 0 = \frac{2}{6^3}$$

Now lets evaluate the denominator. Notice that the *private dice* and *player color* outcome pairs also form a partition of the sample space. Again, using the rule of total probability:

$$P\left((public\ dice = 6) \cap (result = HIT)\right) =$$

$$\sum_{x=green}^{yellow} \sum_{z=1}^{6} P\left((player\ color = x) \cap (public\ dice = 6) \cap (result = HIT) \cap (private\ dice = z)\right) \tag{3}$$

Observe that the inner sum is what we just computed for *player color = blue*. We now simply have to compute the same sum for all the other player colors. To compute the inner sum for a different player color, we have to look at the bottom row (since *public dice = 6*) of the appropriate $6 \times 6$ face on the right side (since *result = HIT*). Then the outer sum adds these values:

$$\underbrace{\frac{1}{6^3}}_{green} + \underbrace{\frac{3}{6^3}}_{purple} + \underbrace{0}_{red} + \underbrace{0}_{orange} + \underbrace{\frac{2}{6^3}}_{blue} + \underbrace{0}_{yellow} = \frac{6}{6^3}$$

Another way to look at the double summation in equation (3) is that this is the sum of the probabilities in the squares on the bottom face of the cube on the right.

Finally, we can substitute our results in to equation (1) to find

$$P\left((player\ color = blue) \mid (public\ dice = 6) \cap (result = HIT)\right) = \frac{2/6^3}{6/6^3} = \frac{2}{6} = \frac{1}{3}$$

This matches with our intuition. You probably guessed $1/3$ for this conditional probability since 2 out of the 6 squares on the bottom row of the game board were blue. Similarly we could have also computed

$$\begin{aligned}
P\left((player\ color = green) \mid (public\ dice = 6) \cap (result = HIT)\right) &= 1/6 \\
P\left((player\ color = purple) \mid (public\ dice = 6) \cap (result = HIT)\right) &= 1/2 \\
P\left((player\ color = red) \mid (public\ dice = 6) \cap (result = HIT)\right) &= 0 \\
P\left((player\ color = orange) \mid (public\ dice = 6) \cap (result = HIT)\right) &= 0 \\
P\left((player\ color = yellow) \mid (public\ dice = 6) \cap (result = HIT)\right) &= 0
\end{aligned}$$

- Notice that the sum of the conditional probabilities for the 6 colors add up to 1.

- If you had to guess the player's color based only on this information, you'd guess **purple** since that has the highest conditional probability. However, the game has more than one round so you could wait to accumulate more information before you make up your mind.

**Question 2:** You are the purple player. Lets assume someone just rolled a 6 in public and announced *HIT*. Based on this information what is the conditional probability that his color is *blue*?

**Solution:** Notice that all the conditional probabilities we have computed are from the view point of an outside observer. If you are a player in the game you know your own color and you know that your opponents can not be that same color! If your own color is purple and your opponent rolled a 6 in public and announced a *HIT*, you should compute the following conditional probability:

$$P\left((player\ color = blue) \mid (public\ dice = 6) \cap (result = HIT) \cap (player\ color \neq purple)\right) =$$
$$\frac{P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = HIT) \cap (player\ color \neq purple)\right)}{P\left((public\ dice = 6) \cap (result = HIT) \cap (player\ color \neq purple)\right)} \quad (4)$$

Notice that the event *player color $\neq$ purple* is the same as $(player\ color = purple)'$. Then notice $(player\ color = blue) \cap (player\ color = purple)' = (player\ color = blue)$. Therefore, the denominator simplifies to

$$P\left((player\ color = blue) \mid (public\ dice = 6) \cap (result = HIT)\right)$$

which we computed as $2/6^3$ before.

The denominator changes because the outer sum now has to avoid the purple color. In other words,

$$P\left((public\ dice = 6) \cap (result = HIT)\right) =$$

$$\sum_x \sum_{z=1}^{6} P\left((player\ color = x) \cap (public\ dice = 6) \cap (result = HIT) \cap (private\ dice = z)\right)$$

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where the sum over $x$ includes all colors except purple. You can find that this is $3/6^3$. Therefore, the conditional probability we are seeking is $\frac{2/6^3}{3/6^3} = 2/3$. Notice that this is higher than the result $1/3$ for the case of the outside observer.

**Question 3:** What if the same player had announced a *MISS* after rolling a 6 for his *public dice*? In this case, we want to compute the following conditional probability:

$$P\left((player\ color = blue)\mid (public\ dice = 6) \cap (result = MISS)\right)$$

From the definition of conditional probability, this is equivalent to:

$$\frac{P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = MISS)\right)}{P\left((public\ dice = 6) \cap (result = MISS)\right)}$$

The only difference from the previous case is now we have to use the plots on the left side corresponding to *MISS*.
Lets evaluate the numerator. Again we use the rule of total probability and the fact that the private dice outcomes form a partition:

$$P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = MISS)\right) =$$

$$\sum_{z=1}^{6} P\left((player\ color = blue) \cap (public\ dice = 6) \cap (result = MISS) \cap (private\ dice = z)\right)$$

From the bottom row of the blue face on the left we see that this resolves as:

$$0 + 0 + \frac{1}{6^3} + \frac{1}{6^3} + \frac{1}{6^3} + \frac{1}{6^3} = \frac{4}{6^3}$$

Now lets evaluate the denominator. Remember that the *private dice* and *player color* outcome pairs also form a partition of the sample space. Using the rule of total probability:

$$P\left((public\ dice = 6) \cap (result = MISS)\right) =$$

$$\sum_{x=green}^{yellow} \sum_{z=1}^{6} P\left((player\ color = x) \cap (public\ dice = 6) \cap (result = MISS) \cap (private\ dice = z)\right) =$$

$$\underbrace{\frac{5}{6^3}}_{green} + \underbrace{\frac{3}{6^3}}_{purple} + \underbrace{\frac{6}{6^3}}_{red} + \underbrace{\frac{6}{6^3}}_{orange} + \underbrace{\frac{4}{6^3}}_{blue} + \underbrace{\frac{6}{6^3}}_{yellow} = \frac{30}{6^3}$$

Therefore:

$$P\left((player\ color = blue) \mid (public\ dice = 6) \cap (result = MISS)\right) = \frac{4/6^3}{30/6^3} = \frac{4}{30} = \frac{2}{15}$$

Similarly we could have also computed

$$
\begin{aligned}
P\left((player\ color = green) \mid (public\ dice = 6) \cap (result = MISS)\right) &= 5/30 \\
P\left((player\ color = purple) \mid (public\ dice = 6) \cap (result = MISS)\right) &= 3/30 \\
P\left((player\ color = red) \mid (public\ dice = 6) \cap (result = MISS)\right) &= 6/30 \\
P\left((player\ color = orange) \mid (public\ dice = 6) \cap (result = MISS)\right) &= 6/30 \\
P\left((player\ color = yellow) \mid (public\ dice = 6) \cap (result = MISS)\right) &= 6/30
\end{aligned}
$$

- Notice that the sum of the conditional probabilities for the 6 colors add up to 1.

- These conditional probabilities are less intuitive to me than the ones for *result = HIT*.

- Its harder to guess the player color based only on this information. In general, the *HIT* results tend to give away more clues, but there is some information to be derived from *MISS* results as well. For instance, if a player rolled 6 in public in a few different turns and always announced *MISS*, you might start to guess he is not likely to be purple.