NAME and UID:

ECE 3530 Midterm 2
Show your work.
No credit will be given for the correct answer if no work is shown.
Four questions each worth 25 points.
Closed book, limited notes (1 regular size sheet front&back). No laptops.

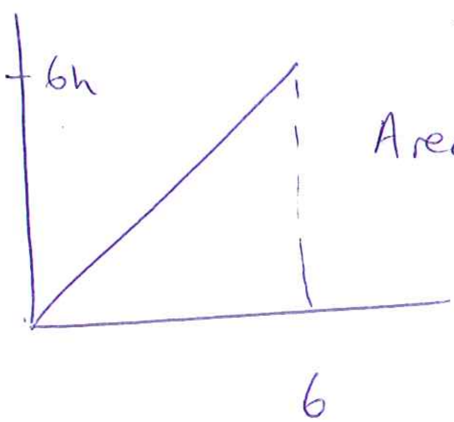
1. A cellphone service provider wants to analyze the signal strengths of its network. Let X be the random variable that is the distance (in miles) that a customer will be to the nearest cellphone tower. Analyzing their database, the company finds that the probability density function for X is

$$f(x) = \begin{cases} hx, & 0 \le x \le 6 \\ 0, & otherwise \end{cases}$$

   (a) Find $P(1 \le X \le 5)$. Give your answer as a numerical value.

   (b) Find the mean and standard deviation of the random variable X.

a)



$$\text{Area} = \frac{36h}{2} = 1 \qquad h = 1/18$$

$$P(1 \le X \le 5) = \int_1^5 \frac{x}{18}\,dx = \frac{x^2}{36}\Big|_1^5$$

$$= \frac{25-1}{36} = 2/3$$

b)
$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^6 x\,\frac{x}{18}\,dx = \frac{x^3}{54}\Big|_0^6 = \frac{6^3}{54} = 4$$

$$\sigma^2 = E[X^2] - \mu^2$$

$$E[X^2] = \int_0^6 x^2 \frac{x}{18}\,dx = \frac{x^4}{18 \times 4}\Big|_0^6 = 18$$

$$\sigma^2 = 18 - 4^2 = 2$$

$$\sigma = \sqrt{2}$$

2. An element has two electrons. Each of the electrons can occupy one of the three orbits of this element. Let the random variables X and Y denote the orbit number occupied by these two electrons. The joint probability distribution is given as:

f(x,y)	x=1	x=2	x=3
y=1	0	1/8	0
y=2	1/8	1/2	1/8
y=3	0	1/8	0

(a) Are these random variables independent?

(b) Find $P(X \leq 2|Y = 2)$.

a) $g(x) = \begin{cases} 1/8, & x=1 \\ 3/4, & x=2 \\ 1/8, & x=3 \end{cases}$ $\quad h(y) = \begin{cases} 1/8, & y=1 \\ 3/4, & y=2 \\ 1/8, & y=3 \end{cases}$

$g(1)\,h(1) = \frac{1}{8}\cdot\frac{1}{8} = \frac{1}{64} = f(1,1) = 0$

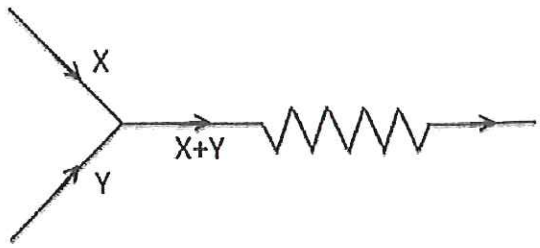
NOT independent

b) $f_X(x|y) = \dfrac{f(x,y)}{h(y)}$

$f_X(x/y=2) = \dfrac{f(x,2)}{h(2)} = \begin{cases} 1/6, & x=1 \\ 2/3, & x=2 \\ 1/6, & x=3 \end{cases}$

$P(X \leq 2|Y=2) = 1/6 + 2/3 = 5/6$

3. Consider the circuit shown below:



The currents X and Y are independent random variables and their marginal densities are given as:

- The current X is a normal (Gaussian) random variable with mean 3 Amperes and standard deviation 0.5 Amperes.
- The current Y has the marginal density function

$$h(y) = \begin{cases} 0.5, & 0 \le y \le 2 \ \textit{Amperes} \\ 0, & \textit{otherwise} \end{cases} \rightarrow \mu_Y = 1$$

(a) Find the probability that the current X is larger than 3.83 Amperes.

(b) The sum of two currents $X+Y$ flow over a into a 10 Ohm resistor. Therefore, the power dissipated over the resistor is $10(X+Y)^2$. Find the average power dissipated over the resistor.

a) $P(X > 3.83) = P\left(Z > \frac{3.83-3}{0.5}\right)$

$= P(Z > 1.66) = 1 - P(Z < 1.66)$

$= 1 - 0.9515 = 0.0485$

b) $E[10(X+Y)^2] = E[10X^2 + 20XY + 10Y^2]$

$= 10E[X^2] + 20E[XY] + 10E[Y^2]$ ⊛

$E[X^2] = \sigma_X^2 + \mu_X^2 = 0.5^2 + 3^2 = 9.25$

independent $E[XY] = E[X]E[Y] = \mu_X \mu_Y = 3 \cdot 1 = 3$

$E[Y^2] = \int_0^2 y^2 \cdot 0.5\, dy = \frac{y^3}{6}\Big|_0^2 = 4/3$

Substitute to ⊛ $10 \times 9.25 + 20 \times 3 + 10 \times 4/3 = 165.83$

4. A random number generator produces binary sequences that are 4-bits long. Each bit can be ON or OFF. Each bit is turned ON with a probability 0.5. The output of a summing circuit is equal to the number of bits that are ON in the 4-bit sequence. Let X be the discrete random variable denoting the output of the summing circuit.

   (a) What is the probability that the ~~brightness level~~ SUM X greater than or equal to 3?

   (b) Another 4-bit random number generator drives another summing circuit whose output we will call Y. Again, each bit is ON with probability 0.5. X and Y are independent random variables. Find the probability that $Y = X + 3$.

a) $P(X \geq 3) = \sum_{x=3}^{4} b(x; n=4, p=0.5)$

$= {}_4C_3\, 0.5^4 + {}_4C_4\, 0.5^4$

$= 5/16$

b) Independence $f(x,y) = g(x)h(y)$

y \ x	0	1	2	3	4
0					
1					
2					
3					
4					

$P(Y = X+3) = $ This Area $= g(0)h(3) + g(1)h(4)$

$= {}_4C_0\, 0.5^4 \cdot {}_4C_3\, 0.5^4 + {}_4C_1\, 0.5^4\, {}_4C_4\, 0.5^4 = \frac{8}{256}$