ECE 3530 HW 6 Solutions

(1)

$f(x,y)$	$x=1$	$x=2$	$x=3$	$h(y)$
$y=1$	0.05	0.05	0.1	0.2
$y=2$	0.05	0.1	0.35	0.5
$y=3$	0	0.2	0.1	0.3
$g(x)$	0.1	0.35	0.55	

a) $$g(x) = \begin{cases} 0.1 & , x=1 \\ 0.35 & , x=2 \\ 0.55 & , x=3 \\ 0 & , \text{otherwise} \end{cases}$$

b) $$h(y) = \begin{cases} 0.2 & , y=1 \\ 0.5 & , y=2 \\ 0.3 & , y=3 \end{cases}$$

c) $P(Y=3 \mid X=2) = f_Y(Y=3 \mid x=2)$

$$f_Y(y \mid x) = \frac{f(x,y)}{g(x)}$$

$$f_Y(y=3 \mid x=2) = \frac{f(2,3)}{g(2)} = \frac{0.2}{0.35} = \frac{4}{7}$$

d) NO.

Take $x=1, y=3$

$f(1,3) = 0 \neq$

$g(1)h(3) = 0.1 \times 0.3$

② a) $P(0<X<1 \mid Y=2) = \int_{x=0}^{1} f_X(x/2)\,dx$

$f_X(x|y) = \dfrac{f(x,y)}{h(y)}$ ⊛

$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} e^{-(x+y)}\,dx = \int_0^{\infty} e^{-x} e^{-y}\,dx$

$= e^{-y} \int_0^{\infty} e^{-x}\,dx = e^{-y}\left[-e^{-x}\right]_0^{\infty} = e^{-y}(-0+1)$

$h(y) = \begin{cases} e^{-y}, & y>0 \\ 0, & \text{elsewhere} \end{cases}$

$f_X(x|y) = \begin{cases} \dfrac{e^{-(x+y)}}{e^{-y}} = e^{-x}, & x>0, y>0 \\ 0, & \text{elsewhere} \end{cases}$

so $P(0<X<1 \mid Y=2) = \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1$

$= -e^{-1} + 1$

$= 1 - \dfrac{1}{e}$

$= 0.632$

b) $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-(x+y)}\,dy$

YES $g(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$

$g(x)h(y) = f(x,y)$ for all x,y

3 a) Substitute $x = 0.5$ into $f_Y(y|x)$ to find

$$f_Y(y|0.5) = \begin{cases} \dfrac{1+2y}{2}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y \le 0.5 \mid X = 0.5) = \int_{-\infty}^{0.5} f_Y(y|0.5)\,dy$$

$$= \int_0^{0.5} \frac{1}{2} + y \, dy = \frac{y}{2}\Big|_0^{0.5} + \frac{y^2}{2}\Big|_0^{0.5}$$

$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

b) we need $f(x,y)$

$f(x,y) = g(x)\, f_Y(y|x)$ from defn. of conditional density

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$P\left(\frac{1}{4} \le X \le \frac{1}{2},\ Y \le 1/2\right) = \int_{x=1/4}^{1/2} \int_{y=0}^{1/2} x+y \, dy\, dx$$

$$= \int_{x=1/4}^{1/2} \left[xy + \frac{y^2}{\cancel{y}\;2} \right]_0^{1/2} dx = \int_{1/4}^{1/2} \frac{x}{2} + \frac{1}{\cancel{4}\;8}\, dx$$

$$= \frac{x^2}{4}\Bigg|_{1/4}^{1/2} + \frac{x}{\cancel{4}\;8}\Bigg|_{1/4}^{1/2} = \cancel{\frac{1}{16}} - \frac{1}{64} + \frac{1}{\cancel{8}\;16} - \cancel{\frac{1}{\cancel{16}\;32}}$$

$$= 5\cancel{7}/64$$

c) $\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y$ (*)

$$\mu_X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy = \int_0^1\int_0^1 x(x+y)\, dx\, dy$$

$$= \int_0^1 \frac{x^3}{3}\Bigg|_{x=0}^{x=1} + \frac{x^2 y}{2}\Bigg|_{x=0}^{x=1} dy = \int_0^1 \frac{1}{3} + \frac{y}{2}\, dy$$

$$= \frac{y}{3}\Bigg|_0^1 + \frac{y^2}{4}\Bigg|_0^1 = 7/12$$

Similarly find $\mu_Y = 7/12$

$$E[XY] = \int_{\cancel{\infty}\,0}^{\cancel{\infty}\,1}\int_{\cancel{\infty}\,0}^{\cancel{\infty}\,1} xy(x+y)\, dx\, dy = \int_0^1 \frac{x^3 y}{3}\Bigg|_{x=0}^{x=1} + \frac{x^2 y^2}{2}\Bigg|_{y=0}^{x=1} dy$$

$$= \int_0^1 \frac{y}{3} + \frac{y^2}{2}\, dy = \frac{y^2}{6}\Bigg|_0^1 + \frac{y^3}{6}\Bigg|_0^1 = 1/3$$

substitute into (*) $\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y = \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = \frac{-1}{144}$

④ a)

$f(x,y)$	$x=1$	2	3	4	5	6	$h(y)$
$y=1$	1/12	1/12	1/24	1/24	1/24	1/24	1/3
2	1/12	1/12	1/24	1/24	1/24	1/24	1/3
3	1/48	1/48	1/96	1/96	1/96	1/96	1/12
4	1/48	1/48	1/96	1/96	1/96	1/96	1/12
5	1/48	1/48	1/96	1/96	1/96	1/96	1/12
6	1/48	1/48	1/96	1/96	1/96	1/96	1/12
$g(x)$	1/4	1/4	1/8	1/8	1/8	1/8	

$f(x,y) = g(x)h(y)$ since X, Y independent

b) $X+Y > 8$ Shaded area A

$$P(X+Y > 8) = \sum_{x,y \in A}\sum f(x,y)$$

$$= 10 \times \frac{1}{96} = 5/48$$

~~c) P(X+Y > 8 | X = 5) = P(Y > 3 | X = 5)~~

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a) $\rho_{XY} = \dfrac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$, $\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y$

$f(x,y)$		x = 0	1	$h(y)$
y	0	0.1	0	0.1
	1	0.1	0.1	0.2
	2	0.1	0.2	0.3
	3	0	0.4	0.4
$g(x)$		0.3	0.7	

$\mu_X = \sum x\, g(x)$
$= 0\times 0.3 + 1\times 0.7$
$= \boxed{0.7}$

$\mu_Y = \sum y\, h(y)$
$= 0\times 0.1 + 1\times 0.2 + 2\times 0.3 + 3\times 0.4$
$= \boxed{2}$

$\sigma_X^2 = E[X^2] - \mu_X^2 = \sum x^2 g(x) - 0.7^2$
$= 0\times 0.3 + 1\times 0.7 - 0.7^2 = 0.7 - 0.49 = \boxed{0.21}$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = \sum y^2 h(y) - 2^2$
$= 0\times 0.1 + 1\times 0.2 + 4\times 0.3 + 9\times 0.4 - 4$
$= 0.2 + 1.2 + 3.6 - 4 = \boxed{1}$

$E[XY] = \sum\sum xy\, f(x,y) = \cancel{0\times 0\times 0.1} + \cancel{1\times 0\times 0}$
$+ \cancel{0\times 1\times 0.1} + 1\times 1\times 0.1 + \cancel{0\times 2\times 0.1}$
$+ 1\times 2\times 0.2 + \cancel{0\times 3\times 0} + 1\times 3\times 0.4$
$= \cancel{0.2}\ 0.1 + 0.4 + 1.2 = \cancel{1.8}\ 1.7$

$\sigma_{XY}^2 = \cancel{1.8}\ 1.7 - 2\times 0.7 = \boxed{\cancel{0.4}}\ 0.3$

$\rho_{XY} = \cancel{0.4}\ 0.3 / \sqrt{0.21}\sqrt{1} = \cancel{0.873}\ 0.65$

b) $P(Y \geq 2, X = 0) = 0.1 + 0 = 0.1$

c) $P(Y \geq 2 \mid X = 0) = f_Y(2|0) + f_Y(3|0)$

$$= \frac{f(0,2)}{g(0)} + \frac{f(0,3)}{g(0)} = \frac{0.1}{0.3} + \frac{0}{0.3}$$

$$= 1/3$$

d) let $u(x,y) = 100x + 10y^2$

Make a table for u

$u(x,y)$	$x = 0$	$x = 1$
$y = 0$	0	100
$y = 1$	10	110
$y = 2$	40	140
$y = 3$	90	190

multiply this table by the table for $f(x,y)$ and sum to compute $E[u(x,y)] = \sum\sum u(x,y) f(x,y)$

$$E[u] = 0 \times 0.1 + 100 \times 0 + 10 \times 0.1 + 110 \times 0.1$$

$$+ 40 \times 0.1 + 140 \times 0.2 + 90 \times 0 + 190 \times 0.4$$

$$= 1 + 11 + 4 + 28 + 76 = 120$$

e) $\cancel{\text{}}$ $E[2Y-4Z] = 2\mu_Y - 4\mu_Z$

$= 2 \times 2 - 4 \times 1 = \boxed{0}$

$var[2Y-4Z] = 2^2 \sigma_Y^2 + (-4)^2 \sigma_Z^2$ since Y, Z independent

$= 4 \times 1 + 16 \times 0.5 = \boxed{12}$