ECE 3530 HW 4 Solution

1. X: discrete, Y: continuous, M: continuous, N: discrete, P: discrete, Q: continuous.

2. a) $f(x) < 0$ for some x such as $x = 1$

   b) $F(x)$ is not monotonically increasing. For instance $F(\pi) < F(\pi/2)$

   c) $F(\infty) = 2 - \frac{1}{\infty} = 2 \neq 1$

   d) It is a valid prob. mass function because $f(x) \geq 0$ for all x and $\sum_x f(x) = 1$

   e) We need $2k + 0.5 + 3k = 1$

   So $5k = 0.5$

   $\boxed{k = 0.1}$

3. a) $\mu = 20 \times 0.2 + 40 \times 0.7 + 60 \times 0.1$
   $= 4 + 28 + 6 = \$38$

   b) $\mu_X = \int_{-\infty}^{\infty} x f(x) dx = \int_{40}^{60} x \frac{1}{40} dx + \int_{60}^{100} x\left(\frac{1}{40} - \frac{x-60}{1600}\right) dx$

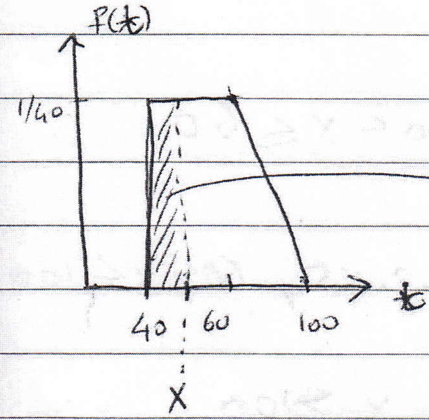
   $= \left.\frac{x^2}{80}\right|_{40}^{60} + \int_{60}^{100} x\left(\frac{1}{40} + \frac{60}{1600}\right) dx - \int_{60}^{100} \frac{x^2}{1600} dx$

   $= 61.667$

c) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Case 1: $x < 40$ $\quad F(x) = \int_{-\infty}^{x} 0\,dt = 0$

Case 2: $40 \le x < 60$ $\quad F(x) = \cancel{\int_{-\infty}^{40} 0\,dt} + \int_{40}^{x} \frac{1}{40}\,dt$

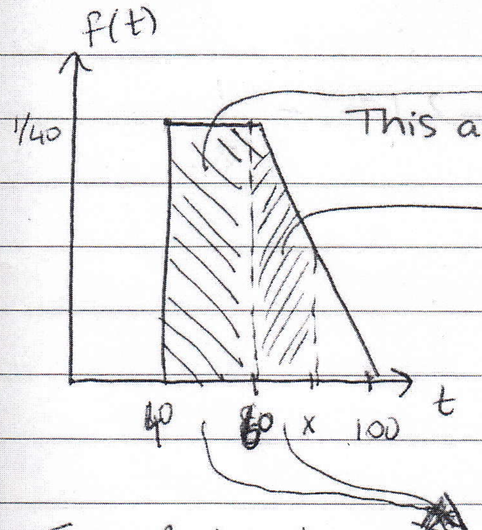


$$= \frac{t}{40}\Big|_{40}^{x} = \frac{x-40}{40}$$

The shaded area is $\frac{1}{40}(x-40)$.

Visual inspection can be faster than integration in cases where f(x) is simple.

Case 3: $60 \le x < 100$ $\quad F(x) = \cancel{\int_{-\infty}^{40} 0\,dt} + \int_{40}^{60} \frac{1}{40}\,dt + \int_{60}^{x} 0.0625 - \frac{t}{1600}\,dt$



This area $= \frac{1}{40} \times (60-40) = 1/2$

$$\int_{60}^{x} 0.0625 - \frac{t}{1600}\,dt = 0.0625t\Big|_{60}^{x} - \frac{t^2}{3200}\Big|_{60}^{x}$$

$$= 0.0625x - 3.75 - \frac{x^2}{3200} + \underbrace{\frac{3600}{3200}}_{1.125}$$

Sum of the two areas $= 0.0625x - \frac{x^2}{3200} - 2.125$

This better be 1 at F(100). Lets check

$$F(100) = 0.0625 \times 100 - \frac{100^2}{3200} - 2.125 = 1$$

So

$$F(x) = \begin{cases} 0, & x < 40 \\ \frac{x-40}{40}, & 40 \le x \le 60 \\ 0.0625x - \frac{x^2}{3200} - 2.125, & 60 \le x \le 100 \\ 1, & x \ge 100 \end{cases}$$

d) $P(X<60) = F(60) = \frac{60-40}{40} = \frac{1}{2}$

e) $P(60<X<80) = F(80) - F(60)$

$$= 0.0625 \times 80 - \frac{80^2}{3200} - 2.125 - \frac{1}{2}$$

$$= 0.375$$

4. a) $P(X \le 1/3) = \int_0^{1/3} 2(1-x)\,dx = 5/9$

b) $P(X \geq 0.5) = \int_{0.5}^{\infty} f(x)\,dx$

$= \int_{0.5}^{1} 2(1-x)\,dx = 1/4$

c) $P(X < 0.75 \mid X \geq 0.5) =$

$$\frac{P(X < 0.75 \cap X \geq 0.5)}{P(X \geq 0.5)}$$

$$= \frac{P(0.5 \le X < 0.75)}{P(X \geq 0.5)} = \frac{\int_{0.5}^{0.75} 2(1-x)\,dx}{1/4}$$

$= 3/4$

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5. a) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_2^6 x \frac{1}{4}\,dx = \frac{x^2}{8}\Big|_2^6$

$$= \frac{36-4}{8}$$

$$= 4$$

$\sigma^2 = E[X^2] - \mu^2$

$$= \int_2^6 x^2 \frac{1}{4}\,dx - 4^2 = \frac{x^3}{12}\Big|_2^6 - 16$$

$$= \frac{6\times6\times6}{12} - \frac{2\times2\times2}{12} - 16 = \frac{4}{3}$$

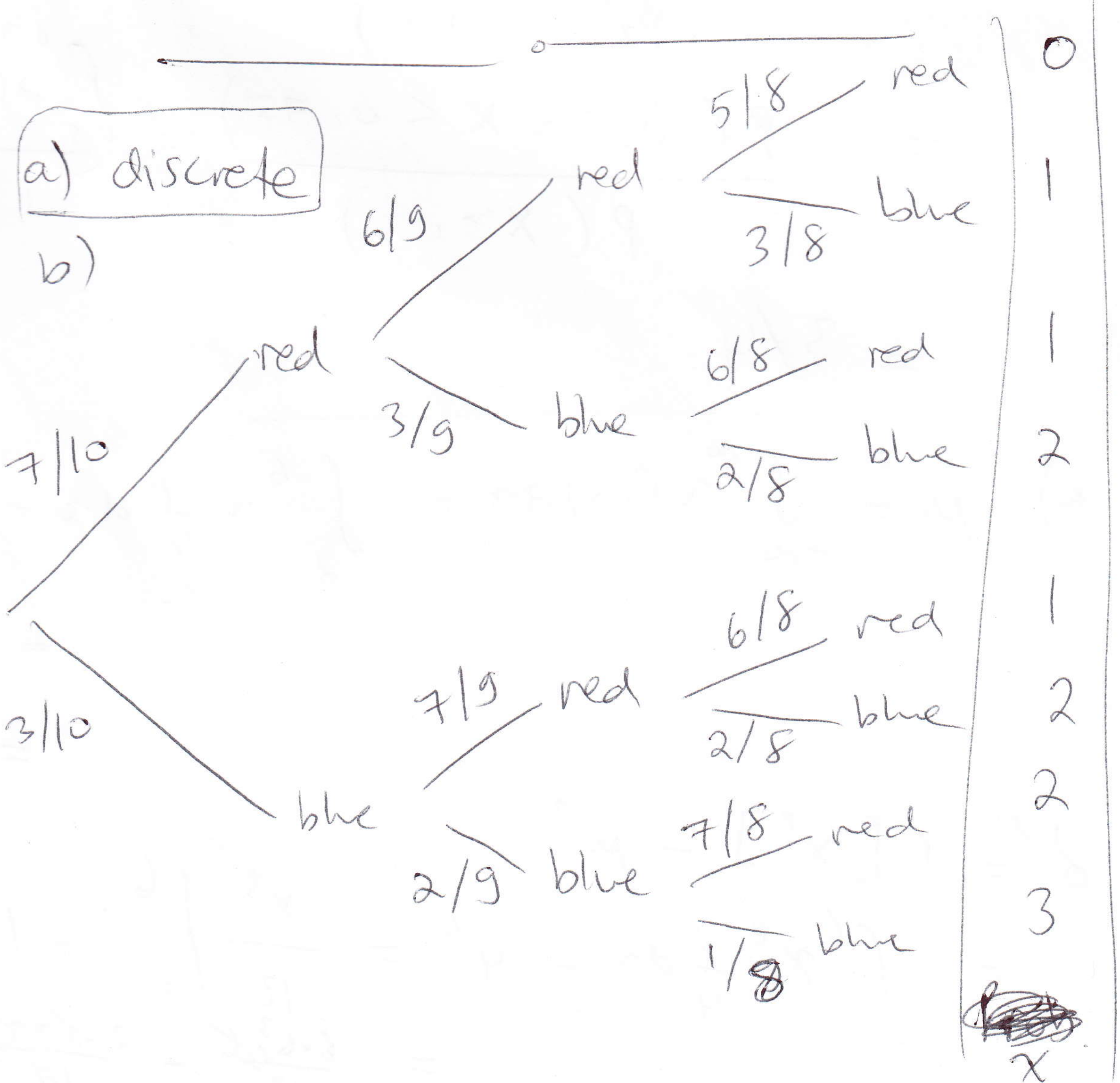
b) $E[P(x)] = E[40/x^2] = \int_2^6 \frac{40}{x^2} \cdot \frac{1}{4}\, dx$

$$= -\frac{10}{x}\Big|_2^6 = -\frac{10}{6} - \left(-\frac{10}{2}\right)$$

$$= 3.33$$

6 a) $\boxed{\text{discrete}}$

b)



$$f(0) = \frac{7}{10} \times \frac{\cancel{6}^{2}}{\cancel{9}_{3}} \times \frac{5}{\cancel{8}_{4}} = \frac{35}{120}$$

$$f(1) = \frac{7}{\cancel{10}_{5}} \times \frac{\cancel{6}^{2}}{\cancel{9}_{3}} \times \frac{3}{8} + \frac{7}{10} \times \frac{3}{9} \times \frac{6}{8}$$

$$+ \frac{3}{10} \times \frac{7}{9} \times \frac{6}{8} = \frac{21}{40}$$

$$f(2) = \frac{7}{10} \times \frac{\cancel{3}}{\cancel{9}_{3}} \times \frac{\cancel{2}}{\cancel{8}_{4}} + \frac{3}{10} \times \frac{7}{9} \times \frac{2}{8}$$

$$+ \frac{3}{10} \times \frac{2}{9} \times \frac{7}{8} = \frac{7}{40}$$

$$f(3) = \frac{\cancel{3}}{\cancel{10}_{5}} \times \frac{\cancel{2}^{1}}{\cancel{9}_{3}} \times \frac{1}{8} = \frac{1}{120}$$

$$f(x) = \begin{cases} 35/120 & , x = 0 \\ 21/40 & , x = 1 \\ 7/40 & , x = 2 \\ 1/120 & , x = 3 \\ 0 & , \text{otherwise} \end{cases}$$

prob-
mass
function

$$F(x) = \sum_{t \leq x} f(t)$$

$$F(x) = \begin{cases} 0 & , x < 0 \\ 35/120 & , x = 0 \\ 98/120 & , x = 1 \\ 119/120 & , x = 2 \\ 1 & , x > 2 \end{cases}$$

c) $P(X \leq 1) = F(1) = 98/120$

d) $\mu_X = \sum x f(x) = 0 \times \frac{35}{120} + 1 \times \frac{21}{40} + 2 \times \frac{7}{40} + 3 \times \frac{1}{120}$

$$= \frac{108}{120}$$

$\sigma^2 = E[X^2] - \mu_X^2$ ⊛

$E[X^2] = 0^2 \times \frac{35}{120} + 1^2 \times \frac{21}{40} + 2^2 \times \frac{7}{40} + 3^2 \times \frac{1}{120}$

$$= \frac{156}{120}$$

⊛ $\sigma^2 = \frac{156 - 108}{120} = \frac{48}{120} = 0.4$

$\sigma = \sqrt{\frac{48}{120}} = 0.632$

e) $P(X) = 100^X$ $\quad P(0) = 1 \quad P(1) = 100$

$P(2) = 10{,}000 \quad P(3) = 1{,}000{,}000$

$E[P(X)] = 1 \times \frac{35}{120} + 100 \times \frac{21}{40} + 10000 \times \frac{7}{40} + \frac{1000000}{120}$

$= \$10{,}136$