Midterm 2 solutions

① a) $F(x) = \sum_{t \le x} f(t)$, so $F(x) = \begin{cases} 0, & x \le 0 \\ 1/6, & x = 1 \\ 2/6, & x = 2 \\ 3/6, & x = 3 \\ 4/6, & x = 4 \\ 5/6, & x = 5 \\ 1, & x \ge 6 \end{cases}$

b) X, Y independent $\longrightarrow$ covariance $\sigma_{XY}^2 = 0$

c) Area has to be 1 so $\frac{2}{5}(k-0) + \frac{4}{5}(2-k) = 1$

$$\frac{2k}{5} + \frac{8}{5} - \frac{4k}{5} = 1$$

$$-\frac{2k}{5} = -3/5 \Rightarrow \boxed{k = 3/2}$$

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② a) First we need to find $f_Y(y|x) = \dfrac{f(x,y)}{g(x)}$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_1^2 \frac{x+y+1}{5}\,dy = \frac{1}{5}\left(xy\Big|_{y=1}^{2} + \frac{y^2}{2}\Big|_{y=1}^{2} + y\Big|_{y=1}^{2} \right)$$

$$= \frac{1}{5}\left(2x - x + 2 - \frac{1}{2} + 2 - 1\right) = \frac{x}{5} + \frac{1}{2} = \frac{2x+5}{10}$$

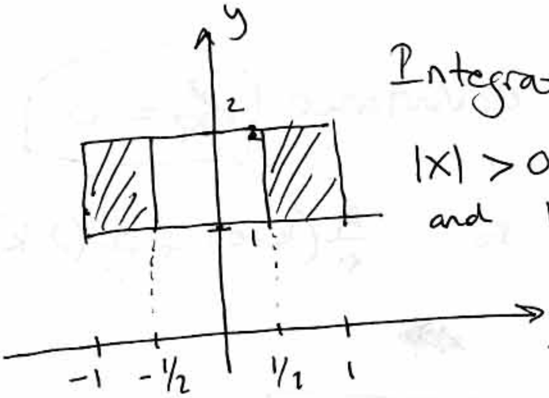
$$f_Y(y|x) = \begin{cases} \dfrac{x+y+1}{5} \cdot \dfrac{10}{2x+5}, & -1 \le x \le 1;\ 1 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y|x=0) = \begin{cases} \dfrac{2(y+1)}{5}, & 1 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y \le 1.5 \mid X=0) = \int_{-\infty}^{1.5} f_Y(y \mid x=0)\,dy = \int_{1}^{1.5} \frac{2}{5}(y+1)\,dy$$

$$= \frac{2}{5}\left(\frac{y^2}{2}\Big|_1^{1.5} + y\Big|_1^{1.5} \right) = \boxed{0.45}$$

b)



Integrate $f(x,y)$ over shaded region $|x| > 0.5$ and $1 \le Y \le 2$ but this is the same as

$$1 - P(|X| \le 0.5, 1 \le Y \le 2)$$

$$= 1 - P(-0.5 \le X \le 0.5, 1 \le Y \le 2)$$

$$= 1 - \int_{y=1}^{2} \int_{x=-0.5}^{0.5} \frac{x+y+1}{5}\,dx\,dy$$

$$= 1 - \frac{1}{5} \int_{y=1}^{2} \left(\frac{x^2}{2}\Big|_{x=-0.5}^{0.5} + xy\Big|_{x=-0.5}^{x=0.5} + x\Big|_{x=-0.5}^{x=0.5} \right) dy$$

$$= 1 - \frac{1}{5} \int_1^2 (y+1)\,dy = 1 - \frac{1}{5}\left(\frac{y^2}{2}\Big|_1^2 + y\Big|_1^2 \right)$$

$$= 1 - \frac{1}{5}\left(2 - 1/2 + 2 - 1 \right) = \boxed{1/2}$$

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3) a) $P(X \le 105) = P\left(Z \le \frac{105 - \mu_X}{\sigma_X}\right)$

$$= P\left(Z \le \frac{105 - 100}{\sqrt{4}}\right)$$

$$= P(Z \le 2.5) = 0.9938 \text{ from Table}$$

b) $Z = \frac{X+Y}{2} = \frac{1}{2}X + \frac{1}{2}Y$

We know $\mu_X = 100$, $\sigma_X^2 = 4$

We need μ_Y and σ_Y^2

$$\mu_Y = \int_{97}^{103} y \frac{1}{103-97}\,dy$$

$$= \frac{1}{6}\left(\frac{y^2}{2} \Big|_{97}^{103} \right) = \frac{1}{12}\left(103^2 - 97^2\right) = 100$$

Could also use the formula $\mu_Y = \frac{103+97}{2} = 100$
for uniform distribution

$\sigma_Y^2 = E[Y^2] - \mu_Y^2$ so we need $E[Y^2]$

$$E[Y^2] = \int_{97}^{103} y^2 \frac{1}{103-97}\,dy = \frac{1}{6}\left(\frac{y^3}{3}\Big|_{97}^{103} \right)$$

$$= \frac{1}{18}\left(103^3 - 97^3\right) = 10003$$

so $\sigma_Y^2 = 10003 - 100^2 = 3$ (or use formula $\sigma_Y^2 = \frac{(103-97)^2}{12}$)

$$\mu_Z = \frac{1}{2}\mu_X + \frac{1}{2}\mu_Y = \frac{1}{2} \times 100 + \frac{1}{2} \times 100 = \boxed{100}$$

$$\sigma_Z^2 = \left(\frac{1}{2}\right)^2 \sigma_X^2 + \left(\frac{1}{2}\right)^2 \sigma_Y^2 = \frac{1}{4} \times 4 + \frac{1}{4} \times 3 = \boxed{\frac{7}{4}}$$

Note: X and Y are independent since the resistors are produced at different factories.

(4) a)

$F(x,y)$	x = 1	2	3	4	$h(y)$
y = 1	0	1/16	1/16	0	1/8
2	1/16	1/8	1/8	1/16	3/8
3	1/16	1/8	1/8	1/16	3/8
4	0	1/16	1/16	0	1/8
$g(x)$	1/8	3/8	3/8	1/8	

Compute $g(x)$ and $h(y)$ by summing over columns and rows, respectively.

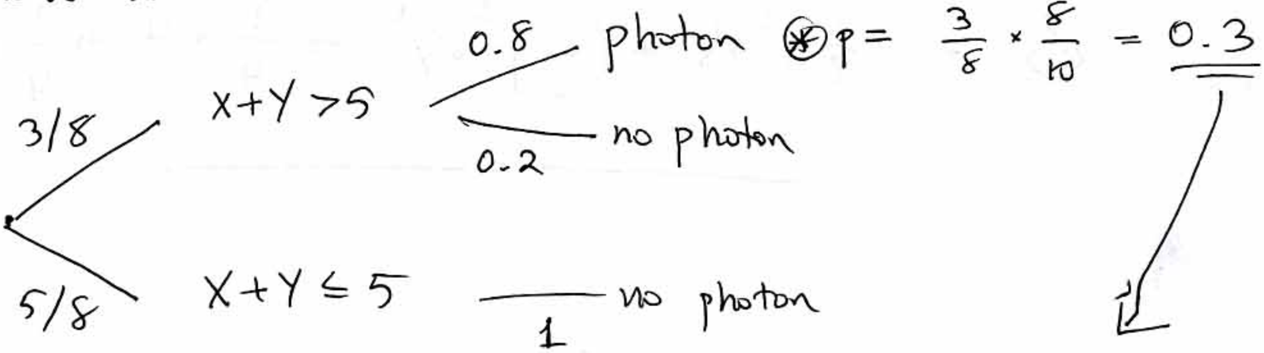
Notice $f(0,0) = 0 \neq g(0)h(0) = \frac{1}{8} \times \frac{1}{8}$

So X, Y not independent.

b) First find $P(X+Y>5)$ by summing over the correct area of the table.

$$P(X+Y>5) = 1/16 + 1/8 + 1/16 + 1/16 + 1/16 + 0 = \frac{3}{8}$$

Now use a tree



6 atoms. Binomial dist with $n=6$, $p=0.3$

$$P(3 \text{ photons}) = {}_6C_3 \cdot 0.3^3 \cdot (1-0.3)^3$$
$$= 0.18522$$