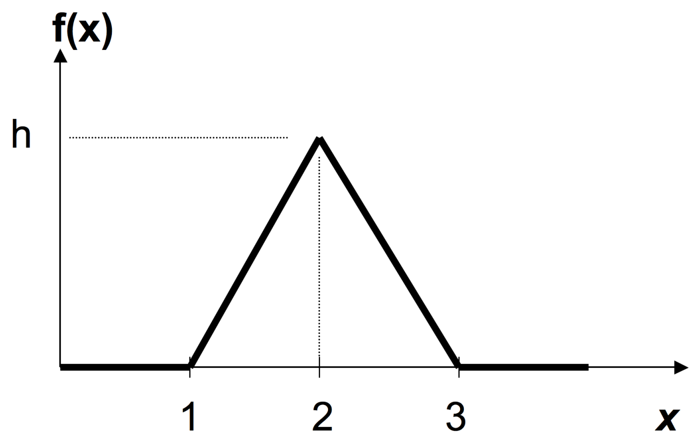
ECE 3530 PRACTICE MIDTERM 2 SOLUTIONS

1. A continuous random variable X has the probability density function:

$$f(x) = \begin{cases} 0, & x < 1 \\ hx - h, & 1 \leq x \leq 2 \\ 3h - hx, & 2 \leq x \leq 3 \\ 0, & x > 3 \end{cases}$$

which can be graphed as



(a) **Find h which makes $f(x)$ a valid probability density function.**
The area underneath the triangle is $\frac{1}{2}(3-1)h = h$ which must equal 1 to satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$. Therefore $h = 1$. You can of course solve this problem by explicitly integrating the function $f(x)$ but it is much simpler to use geometry and the area of the triangle.

(b) **Find the cumulative distribution function $F(x)$.**
First write down the density function after substituting $h = 1$:

$$f(x) = \begin{cases} 0, & x < 1 \\ x - 1, & 1 \leq x \leq 2 \\ 3 - x, & 2 \leq x \leq 3 \\ 0, & x > 3 \end{cases}$$

There are 4 cases:

- $x < 1$: Here $F(x) = 0$.
- $1 \leq x \leq 2$:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{1} 0dt + \int_{1}^{x} (t-1)dt \end{aligned}$$

$$\begin{aligned} &= \left.\frac{t^2}{2}\right|_1^x - t|_1^x \\ &= \frac{x^2}{2} - \frac{1}{2} - x + 1 = \frac{x^2}{2} - x + \frac{1}{2} \end{aligned}$$

- $2 \leq x \leq 3$:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{1} 0dt + \int_1^2 (t-1)dt + \int_2^x (3-t)dt \\ &= F(2) + \int_2^x (3-t)dt \\ &= \left(\frac{2^2}{2} - 2 + \frac{1}{2}\right) + 3t|_2^x - \left.\frac{t^2}{2}\right|_2^x \\ &= \frac{1}{2} + 3x - 6 - \frac{x^2}{2} + 2 = -\frac{x^2}{2} + 3x - \frac{7}{2} \end{aligned}$$

  Notice that $F(3) = 1$ which is what is required.

- $x > 3$: Here $F(x) = 1$

So

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{x^2}{2} - x + \frac{1}{2}, & 1 \leq x \leq 2 \\ -\frac{x^2}{2} + 3x - \frac{7}{2}, & 2 \leq x \leq 3 \\ 1, & x > 3 \end{cases}$$

2. Random variable X has a normal probability distribution with mean 10.3 and standard deviation 2.

   (a) **Compute the numerical value of $P(7.2 \leq X \leq 13.8)$.**
   We first convert to a standard normal distribution with $Z = \frac{X-10.3}{2}$. When $X = 7.2$, $Z = \frac{7.2-10.3}{2} = -1.55$ and $X = 13.8$, $Z = \frac{13.8-10.3}{2} = 1.75$. Therefore,

   $$P(7.2 \leq X \leq 13.8) = P(-1.55 \leq Z \leq 1.75) = P(Z \leq 1.75) - P(Z \leq -1.55)$$

   From the attached table we find that $P(Z \leq 1.75) = 0.9599$ and $P(Z \leq -1.55) = 0.0606$. So the answer is $0.9599 - 0.0606 = 0.8993$.

   (b) **Find a value d such that X is in the range $10.3 \pm d$ with probability 0.999.**
   We want $P(10.3-d \leq X \leq 10.3+d) = 0.999$. Again converting to standard normal distribution: when $X = 10.3 - d$, $Z = \frac{10.3-d-10.3}{2} = -0.5d$ and when $X = 10.3 + d$, $Z = \frac{10.3+d-10.3}{2} = 0.5d$. So we are looking for a d such that $P(-0.5d \leq Z \leq 0.5d) = 0.999$:

   $$\begin{aligned} P(-0.5d \leq Z \leq 0.5d) &= 1 - (P(Z < -0.5d) + P(Z > 0.5d)) \\ 0.999 &= 1 - 2P(Z < -0.5d) \\ P(Z < -0.5d) &= = \frac{1 - 0.999}{2} = 0.0005 \end{aligned}$$

From the attached table we find that $-0.5d = -3.3$, so $d = 6.6$.

(c) **Let Y be a random variable with variance $\sigma_Y^2 = 6$ and independent of X. Compute the variance of $5X - 3Y$.**

$5X - 3Y$ is a linear combination of the random variables X and Y. The variance of X is 4 (the square of its standard deviation). Using the fact that X and Y are independent we find that

$$\sigma_{5X-3Y}^2 = 25\sigma_X^2 + 9\sigma_Y^2 = 25 \times 4 + 9 \times 6 = 154$$

3. Let X and Y be two continuous random variables with the joint density function

$$f(x, y) = \begin{cases} x + y, & 0 \leq x \leq 1 \textit{ and } 0 \leq y \leq 1 \\ 0, & \textit{elsewhere} \end{cases}$$

(a) **Are the random variables X and Y independent? Justify your answer.**

Lets compute the marginal density functions:

$$\begin{aligned} g(x) &= \int_{y=-\infty}^{\infty} f(x, y) dy \\ &= \int_{y=0}^{y=1} (x + y) dy \\ &= xy\Big|_{y=0}^{y=1} + \frac{y^2}{2}\Big|_{y=0}^{y=1} \\ &= x + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} h(y) &= \int x = -\infty^{\infty f}(x, y) dx \\ &= \int_{x=0}^{x=1} (x + y) dx \\ &= \frac{x^2}{2}\Big|_{x=0}^{x=1} + xy\Big|_{x=0}^{x=1} \\ &= \frac{1}{2} + y \end{aligned}$$

For independence we must have $f(x, y) = g(x)h(y)$, which doesn't hold in this case. So X and Y are NOT independent.

(b) **Compute the numerical value of $P(Y \geq \frac{1}{2}, X \leq \frac{1}{2})$.**

$$\begin{aligned} P(Y \geq 1/2, X \leq 1/2) &= \int_{y=1/2}^{y=1} \int_{x=0}^{x=1/2} (x + y) dx dy \\ &= \int_{y=1/2}^{y=1} \left(\frac{x^2}{2}\Big|_{x=0}^{x=1/2} + xy\Big|_{x=0}^{x=1/2} \right) dy \end{aligned}$$

$$= \int_{y=1/2}^{y=1} \frac{1}{8} + \frac{y}{2} dy$$

$$= \left.\frac{y}{8}\right|_{y=1/2}^{y=1} + \left.\frac{y^2}{4}\right|_{y=1/2}^{y=1}$$

$$= \frac{1}{8} - \frac{1}{16} + \frac{1}{4} - \frac{1}{16} = \frac{1}{4}$$

4. Let X be the sent bit and Y the received bit in a binary communications channel. The joint probability distribution $f(x, y)$ is given as:

f(x,y)	x=0	x=1
y=0	0.4	0.1
y=1	0.1	0.4

(a) **Compute the numerical value of** $P(Y = 1|X = 0)$

$$P(Y = 1|X = 0) = \frac{f(0,1)}{g(0)} = \frac{0.1}{0.4 + 0.1} = 0.2$$

(b) **Compute the covariance of random variables** X**,** Y**.**
$\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y$; therefore we first need to compute μ_X, μ_Y and $E[XY]$.

$$\mu_X = 0 \times 0.4 + 0 \times 0.1 + 1 \times 0.1 + 1 \times 0.4 = 0.5$$

$$\mu_Y = 0 \times 0.4 + 1 \times 0.1 + 0 \times 0.1 + 1 \times 0.4 = 0.5$$

$$E[XY] = 0 \times 0.4 + 0 \times 0.1 + 0 \times 0.1 + 1 \times 0.4 = 0.4$$

Therefore

$$\sigma_{XY}^2 = 0.4 - 0.5 \times 0.5 = 0.15$$

(c) **When a single bit is sent and received, we say that an error has occurred if** $Y \neq X$**. If a 8-bit long message is sent over this communication channel, what is the probability that 1 or less errors will occur?**
First find the probability of making an error when a single bit is sent:

$$P(X \neq Y) = P(X = 0, Y = 1) + P(X = 1, Y = 0) = 0.2$$

Each bit sent is a Bernoulli trial with $P(error) = 0.2$. Then the number of errors when 8 bits are sent follow a Binomial distribution $b(x; p = 0.2, n = 8)$. The probability of one or less errors in 8 bits is found as

$$\sum_{x=0}^{x=1} b(x; p = 0.2, n = 8) =_8 C_0 \times 0.2^0 \times 0.8^8 +_8 C_1 \times 0.2^1 \times 0.8^7 \approx 0.5033$$