ECE 3530 PRACTICE MIDTERM 1 SOLUTIONS

1. There is a deck of 15 cards numbered 1 through 15.

   (a) **If you draw 5 cards from the deck without replacement, what is the probability that your hand will contain the cards 10 and 13?**

   Notice that whether these cards are the first two drawn, last two drawn or whatever other order doesn't matter. Therefore to count the total number of outcomes in the sample space we use combinations: ${}_{15}C_5$. Similarly we will use combinations for counting the number of outcomes in the event described in the question. If we fix 2 of the 5 cards in our hand as the 10 and 13 cards, then we still have 3 spots in our hand we need to fill by choosing from $15 - 2 = 13$ available cards. The number of ways to do this is ${}_{13}C_3$. Therefore the probability of the described event is

   $$\frac{{}_{13}C_3}{{}_{15}C_5} = \frac{13!}{(13-3)! \times 3!} \times \frac{(15-5)! \times 5!}{15!} = \frac{13!}{10! \times 3!} \times \frac{10! \times 5!}{15!} = \frac{5 \times 4}{15 \times 14} = \frac{2}{21}$$

   Alternate solution: This question can be solved by permutations but then the answer is

   $$\frac{{}_{13}P_3 \times_5 C_2 \times 2}{{}_{15}P_5} = \frac{2}{21}$$

   In the above equation, the denominator (total number of outcomes) is computed with permutations. The numerator terms: The first term ${}_{13}P_3$ is the number of ways to order 3 cards out of the remaining 13 when cards 10 and 13 have already been selected. The second term ${}_5C_2$ is the number of ways we can choose the 2 places in the hand of 5 for cards 10 and 13 to go. The last term (multiplication by 2) is necessary because we could swap the places of 10 and 13 in the two places set aside for them.

   (b) **If you draw 2 cards from the deck with replacement, what is the probability that the sum of the values of the 2 cards you draw will be an even number?**

   For the sum of two cards to be even, there are two possible ways: both cards even or both cards odd. Denote by E even cards and O odd cards. Notice there are 7 E and 8 O in the deck. Since the cards are drawn with replacement the first and the second draw are independent, therefore:

   $$\begin{aligned} P(EE) &= \frac{7}{15} \times \frac{7}{15} = \frac{49}{225} \\ P(OO) &= \frac{8}{15} \times \frac{8}{15} = \frac{64}{225} \end{aligned}$$

   Since OO and EE are mutually exclusive, the probability of the event described in the question is the sum of these two probabilities:

   $$\frac{49}{225} + \frac{64}{225} = \frac{113}{225}$$

(c) If you draw one card from the deck, then roll one fair dice (6-sided) and finally flip one fair coin (heads or tails). What is the probability that you get an even numbered card OR heads on the coin toss OR both?

First count the total number of possible outcomes: $15 \times 6 \times 2 = 180$.

Case 1: The # of ways to get an outcome with even numbered card: $7 \times 6 \times 2 = 84$

Case 2: The # of ways to get an outcome with heads: $15 \times 6 \times 1 = 90$

If we sum Case 1 and 2 we count twice the cases with an even numbered card AND heads simultaneously. So we need to find how many times that happens: $7 \times 6 \times 1 = 42$ and subtract it from the sum to get $84 + 90 - 42 = 132$. Then the probability is $132/180 = 11/15$.

Alternate solution: We can solve this question using multiplicative rules and independence instead of counting events.

$$\begin{aligned} P(\mathit{even\ card} \cup \mathit{heads}) &= P(\mathit{even\ card}) + P(\mathit{heads}) - P(\mathit{even\ card} \cap \mathit{heads}) \\ &= \frac{7}{15} + \frac{1}{2} - \frac{7}{15} \times \frac{1}{2} = \frac{11}{15} \end{aligned}$$

2. An online bookseller uses one of four shipping companies to send packages to its customers. Any package can be sent with one and only one of these companies. Define the following events:

   - C_1: the package is shipped with company 1
   - C_2: the package is shipped with company 2
   - C_3: the package is shipped with company 3
   - C_4: the package is shipped with company 4

   The bookseller uses the shipping companies with the following probabilities:

$$P(C_1) = 0.5 \quad P(C_2) = 0.25 \quad P(C_3) = 0.125 \quad P(C_4) = 0.125.$$

   Let X be the event that the package arrives on time at its destination. Depending on the shipping company used, the probability of X varies:

$$P(X|C_1) = 0.85 \quad P(X|C_2) = 0.9 \quad P(X|C_3) = 0.8 \quad P(X|C_4) = 0.8.$$

   (a) Compute the numerical value of $P(C_2 \cup C_3)$.

$$P(C_2 \cup C_3) = P(C_2) + P(C_3) - P(C_2 \cap C_3)$$

   But since C_1, C_2, C_3 and C_4 form a partition they are mutually exclusive which means $C_2 \cap C_3 = \emptyset$. Therefore

$$P(C_2 \cup C_3) = P(C_2) + P(C_3) = 0.25 + 0.125 = 0.375$$

   (b) Given that a package has arrived on time what is the probability that it was shipped with company C_1? In other words, compute the numerical value of $P(C_1|X)$.

   Bayes rule states:

$$\begin{aligned} P(C_1|X) &= \frac{P(X|C_1)P(C_1)}{\sum_{k=1}^{4} P(X|C_k)P(C_k)} \\ &= \frac{0.85 \times 0.5}{0.85 \times 0.5 + 0.9 \times 0.25 + 0.8 \times 0.125 + 0.8 \times 0.125} \\ &= \frac{0.425}{0.425 + 0.225 + 0.1 + 0.1} = \frac{0.425}{0.85} = \frac{1}{2} \end{aligned}$$

   (c) Are the events C_1' and X independent? Justify your answer.

   Notice that we computed $P(C_1|X) = 0.5$ in the previous part. Then $P(C_1'|X) = 1 - 0.5 = 0.5$.

   But the question also gives $P(C_1) = 0.5$. Then $P(C_1') = 1 - P(C_1) = 1 - 0.5 = 0.5$.

   Therefore they are independent because $P(C_1'|X) = P(C_1')$.

3. The sample space of an experiment consists of the following outcomes:

$$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

The following events are defined:

- $X =$ The outcome of the experiment is an even number
- $Y = \{6, 7, 8\}$
- $Z = \{3, 7\}$

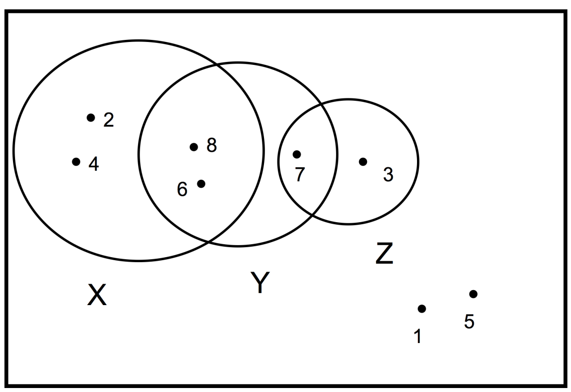
We are also given the following information:

- $P(X \cap Y) = 1/3$
- X and Y are independent events
- Probabilities of all even numbered outcomes are equal. In other words:

$$P(2) = P(4) = P(6) = P(8)$$

Answer the following questions:

(a) Draw the Venn diagram for events X, Y, Z, and all the outcomes in S.



(b) Define a new event W which together with events X and Z forms a partition of the sample space S. Define W by listing the outcomes included in it.

We have to define W to consist of the outcomes in the sample space not included in $X \cup Z$. Therefore $W = \{1, 5\}$.

(c) Compute the numerical value of $P(X \cup Y)$.

From the definition of the probability of an union

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

- You are given $P(X \cap Y) = 1/3$. Notice $X \cap Y = \{6, 8\}$. You are also told all even numbered outcomes are equally likely, therefore, $P(X \cap Y) = P(6) + P(8) =$

$2P(8) = 1/3$ which means $P(8) = P(6) = P(4) = P(2) = 1/6$. The probability of an event is the sum of the probabilities of the outcomes included in it

$$P(X) = P(2) + P(4) + P(6) + P(8) = 4 \times \frac{1}{6} = \frac{4}{6}$$

- To compute $P(Y)$ we use the fact that we are told X and Y are independent events. This means

$$\begin{aligned} P(X \cap Y) &= P(X)P(Y) \\ \frac{1}{3} &= \frac{4}{6}P(Y) \\ \frac{1}{3} \times \frac{6}{4} &= P(Y) \\ \frac{1}{2} &= P(Y) \end{aligned}$$

Substituting these into the equation for $P(X \cup Y)$ we get

$$P(X \cup Y) = \frac{4}{6} + \frac{1}{2} - \frac{1}{3} = \frac{4 + 3 - 2}{6} = \frac{5}{6}$$

4. Two bags A and B each contain a mixture of red balls and black balls. Bag A contains a total of 10 balls of which 5 are red and 5 are black. Bag B contains a total of 16 balls of which 8 are red and 8 are black. In step 1 of a game, a blind folded person chooses one of the two bags with equal probability. Still blind folded, in step 2 of the game he chooses 3 balls without replacement from the bag he chose in the step 1 of the game.

   (a) **What is the probability that he will get 3 red balls in step 2, if he chose bag A in step 1?**

   Notice the only way for this to happen is we get RRR (red ball for each draw). Since the draw is without replacement we use the multiplicative rule:

$$\frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{1}{12}$$

   (b) **If he got 3 red balls in step 2, what is the probability that he chose bag B in step 1?**

   Notice that A and B are separate bags and they are the only two bags available; hence, they form a partition. Lets call X the event that we get 3 red balls when 3 balls are picked. This question is asking $P(B|X)$. Using Bayes rule

$$P(B|X) = \frac{P(X|B)P(B)}{P(X|A)P(A) + P(X|B)P(B)}$$

   We choose bags A and B with equal probability therefore $P(A) = P(B) = 1/2$. We also just computed $P(X|A)$ in part (a). Lets compute $P(X|B)$ in a similar manner:

$$P(X|B) = \frac{8}{16} \times \frac{7}{15} \times \frac{6}{14} = \frac{1}{10}$$

   Substituting these into the equation for $P(B|X)$ we have

$$P(B|X) = \frac{\frac{1}{10} \times \frac{1}{2}}{\frac{1}{12} \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2}} = \frac{\frac{1}{10}}{\frac{1}{12} + \frac{1}{10}} = \frac{1/10}{11/60} = \frac{6}{11}$$