CONTINUOUS UNIFORM DISTRIBUTION

Definition: The probability density function of the continuous uniform random variable \( X \) on the interval \([A, B]\) is

\[
P(x; A, B) = \begin{cases} 
\frac{1}{B-A}, & A \leq x \leq B \\
0, & \text{elsewhere}
\end{cases}
\]

\[
f(x; A, B) = \frac{1}{B-A}
\]

Mean of \( f(x; A, B) \):

\[
\mu = \int_{-\infty}^{\infty} x f(x; A, B) \, dx = \int_{A}^{B} x \frac{1}{B-A} \, dx = \frac{x^2}{2(B-A)} \bigg|_{A}^{B} = \frac{B^2-A^2}{2(B-A)} = \frac{B^2-A^2}{2(B-A)} = \frac{(B-A)(B-A)}{2(B-A)} = \frac{(B+A)/2}{2(B-A)}
\]

Variance of \( P(x; A, B) \):

\[
\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x; A, B) \, dx - \mu^2 = \int_{A}^{B} \frac{x^2}{B-A} \, dx - \left( \frac{B+A}{2} \right)^2
\]

\[
= \frac{x^3}{3(B-A)} \bigg|_{A}^{B} - \left( \frac{B+A}{2} \right)^2
\]
\[ \frac{B^3 - A^3}{3(B-A)} - \frac{(B+A)^2}{4} \]
\[ = \frac{(B-A)(B^2 + AB + A^2)}{3(B-A)} - \frac{(B+A)^2}{4} \]
\[ = \frac{4B^2 + 4AB + 4A^2 - 3B^2 - 3AB - 3A^2}{12} \]
\[ = \frac{B^2 - 2AB + A^2}{12} = \frac{(B-A)^2}{12} \]

**Continuous Normal Distribution (a.k.a. Gaussian)**

Described by 2 parameters:
- mean \( \mu \)
- standard deviation \( \sigma \)

**Defn:** The probability density function of the normal (Gaussian) random variable \( X \) with mean \( \mu \) and standard deviation \( \sigma \) is

\[ n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

**Notes:**
- \( (x-\mu)^2 \) is squared distance from the mean
- \( e^{-\frac{(x-\mu)^2}{2\sigma^2}} \) gets smaller as \( (x-\mu)^2 \) gets larger
- How fast it gets small depends on \( \sigma \). Faster for small \( \sigma \)
- The term \( \frac{1}{\sqrt{2\pi} \sigma} \) makes sure \( \int_{-\infty}^{\infty} n(x; \mu, \sigma) \, dx = 1 \)

Many physical phenomena are described very well by a normal distribution:
- Human height
- Measurement errors
- Daily changes in Dow Jones Index
- Many more
Examples a) Two normal distributions with the same variance but different means.

\[ \sigma_1 = \sigma_2 \]

\[ \mu_1 \quad \mu_2 \]

b) Two normal distributions with the same mean but different variances.

\[ \sigma_1 \neq \sigma_2 \]

\[ \sigma_1 \quad \sigma_2 \]

\[ \mu_1 = \mu_2 \]

Observations
- The mode of a density function is where the x value for which \( f(x) \) is highest. For the normal density the mode is equal to the mean.
- The normal distribution is symmetric about \( \mu \).
- The normal curve has points of inflection at \( \mu \pm \sigma \).
- The "" approaches 0 asymptotically as we move away from \( \mu \) in either direction.

Question: How do we know that the mean of \( n(x; \mu, \sigma) \) is really \( \mu \) and the variance is really \( \sigma^2 \)?
Proof of mean

\[ \mu = \mathbb{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \]

let \( z = \frac{x-\mu}{\sigma} \) and \( dx = \sigma \, dz \)

then \( \mu = \mathbb{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{z^2}{2}} \, dz \)

\[ \mathbb{E}[X] = \mu \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz + \sigma \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-\frac{z^2}{2}} \, dz \]

This is the entire area under the normal curve. Hence it equals 1.

\[ \mu \cdot 1 + \sigma \cdot 0 = \mu \]

The proof of variance = \( \sigma^2 \) can be accomplished with a similar change of variables followed by integration by parts. See page 175 of textbook.
Example: A certain type of battery lasts, on average 3 years with a standard deviation of 0.5 years. Assuming battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.

\[ P(X < 2.3) = \int_{-\infty}^{2.3} \frac{1}{\sqrt{2\pi}0.5} e^{-\frac{(x-3)^2}{2(0.5)^2}} \, dx \]

Unfortunately, the difficulty encountered in solving integrals of the normal density function necessitates tabulation of normal curve areas for reference. But we can't make tables for all possible \( \mu, \sigma \)!

Let \( Z = \frac{X - \mu}{\sigma} \)

Notice \( Z \) has 0 mean and variance = 1. It is said to have a standard normal dist.

\[ Z = \frac{2.3 - 3}{0.5} = -1.4 \]

Now \( P(X < 2.3) = P(Z < -1.4) = \int_{-\infty}^{-1.4} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz \)

So we only need table for the standard normal distribution to be able to evaluate integrals of arbitrary normal distributions with any \( \mu, \sigma \).

Table A.3, Textbook

\[ P(Z < -1.4) = 0.0808 \Rightarrow P(X < 2.3) = 0.0808 \]
Example: For the same battery life problem, what is the probability that a given battery will last between 1.5 to 2 years?

\[ z = \frac{X - 3}{0.5} \]

- When \( x = 2 \)
  \[ z = \frac{2 - 3}{0.5} = -2 \]
- When \( x = 1.5 \)
  \[ z = \frac{1.5 - 3}{0.5} = -3 \]

\[
P(1.5 < X < 2) = P(-3 < Z < -2) = P(Z < -2) - P(Z < -3)
\]

From Table A.3.

\[ P(Z < -2) = 0.0228 \quad P(Z < -3) = 0.0013 \]

Therefore

\[ P(1.5 < X < 2) = 0.0228 - 0.0013 = 0.0215 \]

\[
P(3 < X < 4) = ? \quad z = \frac{X - 3}{0.5}
\]

- \( x = 3 \rightarrow z = 0 \)
- \( x = 4 \rightarrow z = 2 \)

\[
P(3 < X < 4) = P(Z < 2) - P(Z < 0) = 0.9772 - 0.5 = 0.4772
\]
Example: A company manufactured electrical resistors with mean 3 Ohms and standard deviation 0.005 Ohms. A buyer sets specifications on the resistance to be 3.0 ± 0.01 Ohms. On the average, what percentage of resistors will be rejected?

Solution:

\[ z = \frac{x - 3}{0.005} \]

\[ x = 2.99 \rightarrow z = -2 \]

\[ x = 3.01 \rightarrow z = 2 \]

First find

\[ P(2.99 < x < 3.01) \]

\[ = P(-2 < z < 2) \]

\[ = P(z < 2) - P(z < -2) = 0.9772 - 0.0228 = 0.9544 \]

Then \( P(\text{reject}) = P(z < -2) + P(z > 2) \)

\[ = 1 - P(-2 < z < 2) \]

\[ = 1 - 0.9544 = 0.0456 \]

Alternative solution: Due to symmetry

\[ P(z < -2) = P(z > 2) \]

Hence \( P(\text{reject}) = P(z < 2) + P(z > 2) \)

\[ = 2 \times P(z < -2) \]

\[ = 2 \times 0.0228 = 0.0456 \]
Example: Using the Normal Curve in reverse

A company manufactured electrical resistors with mean 3 Ohms. The standard deviation is 0.1 Ohms. Find a value of such that 95% of all manufactured resistors fall in the range $3 \pm d$ Ohms.

$$Z = \frac{X - 3}{0.1}$$

From Table A-3 we know that $P(Z \leq -1.96) = 0.025$

Hence $P(-1.96 < Z < 1.96) = 1 - 2 \times 0.025 = 0.95$

Finally we have to convert $Z = 1.96$ back to $X$

$$Z = \frac{X - 3}{0.1} \text{ so } X = 0.1Z + 3$$

$$0.1 \times 1.96 + 3 = 3.196$$

$$0.1 \times -1.96 + 3 = 2.04$$

$P(2.04 < X < 3.196) = 0.95$

$3 \pm 0.196 \Rightarrow d = 0.196$