Means and Variances of Linear Combinations of r.v.s

Theorem: $E[aX+b] = aE[X]+b$

Notation: X random variable. a, b scalars.

Proof:

$$E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$= a\underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{E[X]} + b\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= aE[X]+b$$

Example: Let X be a random variable with expected value (mean) 5. What is the expected value of the functions $u(X) = 3X - 10$ and $v(X) = 1 - 2X$

Soln: $E[3X-10] = 3E[X]-10 = 15-10 = 5$

$E[1-2X] = 1-2E[X] = 1-10 = -9$

Theorem: $E[ag(X)+bh(X)] = aE[g(X)] + bE[h(X)]$

Book Notation: do not confuse these functions $g(X)$, $h(X)$ with the marginal densities.

Our notation: we will use $u(X), v(X)$ instead so:

$$E[au(X)+bv(X)] = aE[u(X)] + bE[v(X)]$$

Proof: $E[au(X)+bv(X)]$

$$= \int_{-\infty}^{\infty} (au(x)+bv(x)) f(x)\,dx = a\underbrace{\int_{-\infty}^{\infty} u(x)f(x)\,dx}_{E[u(X)]} + b\underbrace{\int_{-\infty}^{\infty} v(x)f(x)\,dx}_{E[v(X)]}$$

Form given in the book:

$$a = 1,\ b = 1 \Rightarrow E[u(X) + v(X)] = E[u(X)] + E[v(X)]$$

$$a = 1,\ b = -1 \Rightarrow E[u(X) - v(X)] = E[u(X)] - E[v(X)]$$

<u>Example</u>:
$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Compute $E[X^2 + X - 2]$.

$$E[X^2 + X - 2] = E[X^2] + E[X] - E[2]$$

$$E[2] = \int_{-\infty}^{\infty} 2 f(x)\,dx = 2 \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= 2.$$

expected value of a constant is the same constant

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_1^2 x\, 2(x-1)\,dx = \int_1^2 2x^2 - 2x\,dx$$

$$= \frac{2}{3}x^3\Big|_1^2 - x^2\Big|_1^2 = \frac{16}{3} - \frac{2}{3} - 4 + 1$$

$$= \frac{14}{3} - 3 = 5/3$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_1^2 x^2\, 2(x-1)\,dx = \int_1^2 2x^3 - 2x^2\,dx$$

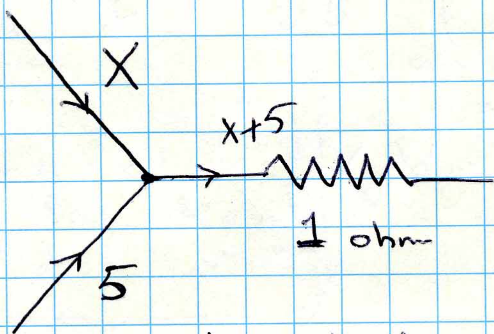
$$= \frac{x^4}{2}\Big|_1^2 - \frac{2x^3}{3}\Big|_1^2 = 8 - \frac{1}{2} - \frac{16}{3} + \frac{2}{3}$$

$$= \frac{15}{2} - \frac{14}{3} = \frac{45-28}{6} = 17/6$$

$$\text{so } E[X^2 + X - 2] = E[X^2] + E[X] - E[2]$$
$$= \frac{17}{6} + \frac{5}{3} - 2 = 5/2$$

Nothing really new in this. We could have computed $E[x^2 + X - 2]$ directly as $\int_1^2 (x^2 + x - 2)\, 2(x-1)\, dx$

<u>Example</u>:



X is a random variable. We know that the average current X is $\mu_X = 1$

We also know the average power dissipated over the 1 ohm resistor is 40 Watts. Find σ_X^2, the variance of the current X.

<u>Hint</u>: power $= I^2 R$ so in this case $(X+5)^2 \cdot 1$

<u>Soln</u>: We are given $E[(X+5)^2] = 40$, $E[X] = 1$

$$E[(X+5)^2] = E[X^2 + 10X + 25] = 40$$
$$= E[X^2] + 10\underbrace{E[X]}_{\text{given as } 1} + 25 = 40$$

$$\text{so } E[X^2] + 35 = 40 \Rightarrow E[X^2] = 5$$

$$\text{Since } \sigma_X^2 = E[X^2] - \mu_X^2 = 5 - E[X]^2 = 5 - 1$$
$$= 4.$$

remember $\mu_X = E[X]$

Theorem

If X, Y independent $E[XY] = E[X]E[Y]$

Proof:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

using independence

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x)h(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} y h(y) \underbrace{\left[\int_{-\infty}^{\infty} x g(x)\,dx\right]}_{\mu_X = E[X]} dy$$

$$= \int_{-\infty}^{\infty} y h(y)\, \mu_X = \mu_X \underbrace{\int_{-\infty}^{\infty} y h(y)\,dy}_{\mu_Y = E[Y]}$$

$$E[XY] = \mu_X \mu_Y = E[X]E[Y]$$ when X, Y independent.

Theorem: If a and b are constants, then

$$\sigma^2_{aX+b} = a^2 \sigma_X^2 .$$

Proof: $\sigma^2_{aX+b} = E\left[\left(aX+b - \mu_{aX+b}\right)^2\right]$

so first compute μ_{aX+b}

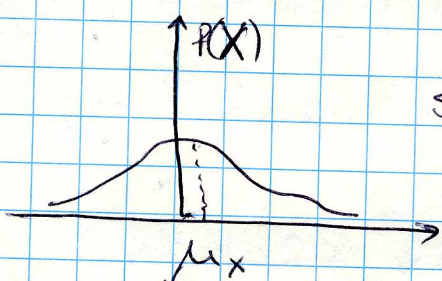
$$\mu_{aX+b} = E[aX+b] = a\mu_X + b.$$

Then
$$\sigma^2_{aX+b} = E\left[\left(aX+\cancel{b} - a\mu_X - \cancel{b}\right)^2\right]$$
$$= E\left[a^2(X-\mu_X)^2\right]$$
$$= a^2 E\left[(X-\mu_X)^2\right] = a^2\sigma_X^2$$

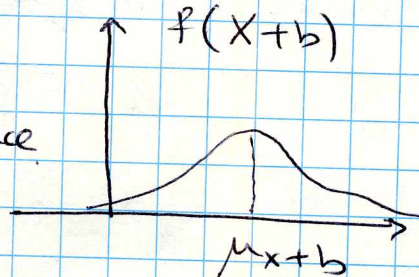
Special cases

(1) $a=1$: $\sigma^2_{X+b} = \sigma_X^2$

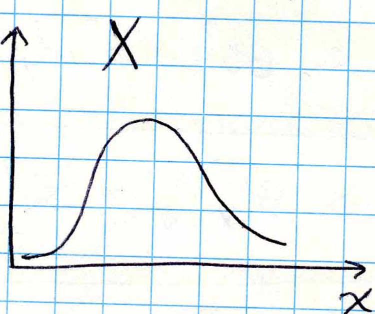
Shifting the density function left or right has no effect on the variance.



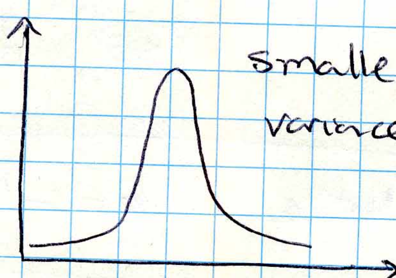
Same variance



(2) $b=0$: $\sigma^2_{aX} = a^2\sigma_X^2$

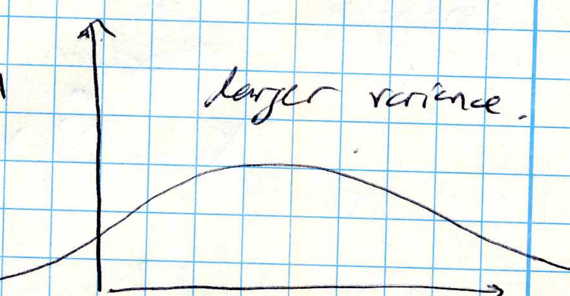


$a<1$



smaller variance

$a>1$



larger variance.

Theorem: X, Y joint distribution $f(x,y)$

$$\sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}^2$$

Proof: First compute μ_{aX+bY}

$$\mu_{aX+bY} = E[aX+bY] = a\mu_X + b\mu_Y$$

then $\sigma_{aX+bY}^2 = E\left[\left(aX+bY-(a\mu_X+b\mu_Y)\right)^2\right]$

$$= E\left[\left(a(X-\mu_X)+b(Y-\mu_Y)\right)^2\right]$$

$$= E\left[a^2(X-\mu_X)^2\right] + E\left[b^2(Y-\mu_Y)^2\right] + E\left[2ab(X-\mu_X)(Y-\mu_Y)\right]$$

$$= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}^2$$

Corollary: If X, Y independent then $\sigma_{XY}^2 = 0$

so $\sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2$

Notice if $b=-1$, $a=1 \Rightarrow \sigma_{X-Y}^2 = \sigma_X^2 + \sigma_Y^2$

Corollary: If $X_1, X_2, \ldots, X_n$ are independent random variables, ~~then~~ and if

$$X = \sum_{k=1}^{n} X_k \text{ then}$$

$$\sigma_X^2 = \sum_{k=1}^{n} \sigma_{X_k}^2$$

Example: X, Y independent random variables with $\sigma_X^2 = 2$ and $\sigma_Y^2 = 3$. Find the variance of the random variable $Z = 3X - 2Y + 5$

$$\sigma_Z^2 = \sigma^2_{(3X-2Y)+5} = \sigma^2_{3X-2Y}$$

$$= \underline{9\sigma_X^2 + 4\sigma_Y^2 = 30.}$$

Mean and Variance of the Binomial Distribution

n repeated Bernoulli trials: $X_1, X_2, \ldots X_n$ each X_n can have two outcomes. Call these "0" and "1" with probabilities p and $q = 1-p$. Then X is the number of "1" outcomes (successes) out of the n trials.

(*) $\mu_X = \mu_{X_1+X_2+\cdots+X_n} = E[X_1 + X_2 + \cdots + X_n]$

$$= E[X_1] + E[X_2] + \cdots + E[X_n]$$

$$= \sum_{k=1}^{n} E[X_k] = np$$

$$\boxed{\begin{aligned} E[X_k] &= \sum x f(x) \\ &= 1 \times p + 0 \times (1-p) \\ &= p. \end{aligned}} \xRightarrow{\text{substitute}}$$

(*) $\sigma_X^2 = \sum_{k=1}^{n} \sigma_{X_k}^2$ (from last corollary)

$$= npq$$

$$= np(1-p)$$

$$\xLeftarrow{\text{substitute}} \boxed{\begin{aligned} \sigma_{X_k}^2 &= \sum (x-\mu)^2 f(x) \\ &= (1-p)^2 p + (0-p)^2 (1-p) \\ &= (1-p)[(1-p)p + p^2] \\ &= (1-p)(p - \cancel{p^2} + \cancel{p^2}) \\ &= (1-p)p = qp \end{aligned}}$$