**Continuous Joint Density Function**

**Defn.** Joint density function \( f(x, y) \) of the continuous random variables \( X \) and \( Y \)

\[
\begin{align*}
&\text{a) } f(x, y) > 0 \text{ for all } (x, y) \\text{ (non-negative)} \\
&\text{b) } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1 \text{ (normalization)}
\end{align*}
\]

Also for any region \( A \) in the \( xy \) plane,

\[
P\left[ (X, Y) \in A \right] = \int_A f(x, y) \, dx \, dy
\]

**Example:**

\[
f(x, y) \text{ shaped like a cone}
\]

\[
f(x, y) = \begin{cases} 
\frac{h(2 - \sqrt{x^2 + y^2})}{2}, & \sqrt{x^2 + y^2} \leq 2 \\
0, & \text{otherwise}
\end{cases}
\]

Find \( h \) that makes \( f(x, y) \) a valid density.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \text{Volume under cone}
\]

\[
= \int_{0}^{h} \left( \text{Area of circle at height } z \right) \, dz
\]

\[
= \int_{0}^{h} 4\pi \left( \frac{h-z}{h} \right)^2 \, dz
\]

\[
\Rightarrow r = \frac{h-z}{h}
\]

Radius of circle at height \( z \)
\[
\text{Vol} = \frac{4\pi}{h^2} \int_0^h (h^2 - 2hz + z^2) \, dz
\]
\[
= \frac{4\pi}{h^2} \left( \left. h^2z \right|_0^h - \left. h^2z \right|_0^h + \left. \frac{z^3}{3} \right|_0^h \right)
\]
\[
= \frac{4\pi}{h^2} \left( h^3 - h^3 + \frac{h^3}{3} \right)
\]
\[
= \frac{4\pi h}{3}
\]

Since \( \text{Vol} = 1 \), \( \frac{4\pi h}{3} = 1 \) \( \Rightarrow h = \frac{3}{4\pi} \)

* Find the probability that \( X^2 + Y^2 \leq 1 \):

\[
P(X^2 + Y^2 \leq 1) = \int_A f(x, y) \, dx \, dy
\]

where \( A \) is the disk \( X^2 + Y^2 \leq 1 \) in the \( xy \) plane

\[
P(X^2 + Y^2 \leq 1) = \int_A f(x, y) \, dx \, dy
\]
Change to polar coordinates $r = \sqrt{x^2 + y^2}$

\[
\iint_A f(x, y) \, dx \, dy = \int_0^{2\pi} \int_0^1 r f(r, \theta) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 r \frac{3}{8\pi} (2 - r) \, dr \, d\theta
\]

\[
= \frac{3}{8\pi} \int_0^{2\pi} \left[ \int_0^1 2r - r^2 \, dr \right] \, d\theta
\]

\[
= \frac{3}{8\pi} \int_0^{2\pi} \left[ r^2 \right]_0^1 - \frac{r^3}{3} \bigg|_0^1 \right] d\theta
\]

\[
= \frac{3}{8\pi} \int_0^{2\pi} \left( 1 - \frac{1}{3} \right) d\theta = \frac{3}{8\pi} \cdot \frac{2\pi}{3} = \frac{1}{4\pi} \cdot 2\pi = \frac{1}{2}
\]

**Defn.** The marginal density functions of $X$ alone and $Y$ alone are

\[g(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y) \, dx\]

**Example:** see Figures for marginal and conditional density examples with the cone shaped density function from previous example.
The areas under the red curves are the values for the marginal distribution $g(x)$ evaluated at $x=-1$, $x=0$ and $x=1$

If we draw the red curves at each value of $x$, and for each compute the area underneath, we get the marginal distribution $g(x)$ which we can then plot as a graph.
Once normalized by dividing with the appropriate value of $g(x)$, the red curves are the conditional densities $f_Y(y|x = -1)$, $f_Y(y|x = 0)$ and $f_Y(y|x = 1)$.

The areas under the green curves are the values for the marginal distribution $h(y)$ evaluated at $y=-1$, $y=0$ and $y=1$. Again if we normalize these curves by dividing with the appropriate value of $h(y)$, the green curves become the conditional densities $f_X(x|y = -1)$, $f_X(x|y = 0)$ and $f_X(x|y = 1)$. 
Example:

\[ f(x, y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \]

\[ V_0 = \frac{1}{\pi} \cdot \pi = 1 \checkmark \]

Compute the marginal densities.

Shaded area is \( g(x_0) \).

Imagine collapsing the \( y \) axis. \( g(x) \) becomes the area under \( P(x, y) \) for any given \( x \).

\[ y = \sqrt{1 - x_0^2} \]

Length of line from \((x_0, -\sqrt{1 - x_0^2})\) to \((x_0, \sqrt{1 - x_0^2})\) is \(2\sqrt{1 - x_0^2}\).

Shaded area is \( \frac{1}{\pi} \cdot 2\sqrt{1 - x_0^2} \).

Therefore \( g(x) = \begin{cases} \frac{2}{\pi} \sqrt{1 - x^2}, & -1 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases} \)

Also due to symmetry \( h(y) = \begin{cases} \frac{2}{\pi} \sqrt{1 - y^2}, & -1 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases} \)
You can check that $g(x)$ is a proper density function. We must have $\int_{-\infty}^{\infty} g(x) \, dx = 1$

\[
\int_{-\infty}^{\infty} g(x) \, dx = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2} \, dx
\]

From an integral table, $\int \sqrt{1-x^2} \, dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\tan^{-1}\frac{x}{\sqrt{1-x^2}}}{2}$

So we have

\[
\frac{2}{\pi} \left( \frac{x\sqrt{1-x^2}}{2} \right)_{-1}^{1} + \frac{\tan^{-1}\frac{x}{\sqrt{1-x^2}}}{2} \bigg|_{-1}^{1}
\]

\[
= \frac{2}{\pi} \left( 0 - 0 + \tan^{-1}\frac{1}{\sqrt{0}} - \tan^{-1}\frac{-1}{\sqrt{0}} \right)
\]

\[
= \frac{2}{\pi} \left( \frac{\pi/2 - (-\pi/2)}{2} \right) = \frac{2}{\pi} \times \frac{\pi}{2} = 1
\]

Plot of the marginal density $g(x)$
Let's compute the conditional density \( f_y(y|x) \) for our example.

**Defn:** The conditional density functions are defined as

\[
\begin{align*}
  f_y(y|x) &= \frac{f(x,y)}{g(x)}, g(x) > 0 \\
  f_x(x|y) &= \frac{f(x,y)}{h(y)}, h(y) > 0
\end{align*}
\]

\[
f(x,y) = \begin{cases} 
  \frac{1}{\pi}, & x^2 + y^2 \leq 1 \\
  0, & \text{elsewhere}
\end{cases}
\]

\[
g(x) = \begin{cases} 
  \frac{2}{\pi} \sqrt{1-x^2}, & -1 \leq x \leq 1 \\
  0, & \text{elsewhere}
\end{cases}
\]

a) From the definition, \( f_y(y|x) \) is defined only for those values of \( x \) for which \( g(x) > 0 \). For an example this is \(-1 \leq x \leq 1\).

b) Once we fix a particular \( x \), \( f_y(y|x) \) at that \( x \) is the cross-section of \( f(x,y) \) at that \( x \) normalized by the area underneath the cross-section which is \( g(x) \). With \( x \) fixed, we can have the formula for the cross-section

\[
\frac{1}{\pi} \text{ for } y^2 \leq 1 - x^2 \\
0 \text{ otherwise.}
\]
The function $f_y(y | x) = \begin{cases} \frac{1}{2\sqrt{1-x^2}}, & y^2 \leq 1-x^2 \text{ and } -1 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

From this formula:

$$f_y(y | -0.5) = \begin{cases} \frac{1}{2\sqrt{0.75}}, & -\sqrt{0.75} \leq y \leq \sqrt{0.75} \\ 0, & \text{otherwise} \end{cases}$$

$$f_y(y | -0.5) \approx 0.577$$
At $x = 0$,

$$f_Y(y|1) = \begin{cases} \frac{1}{2}, & -1 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Notice that this graph is wider and lower than the graph for $f_Y(y|1 - 0.5)$.

At $x = 0.5$,

$$f_Y(y|0.5) = \begin{cases} 1.147, & -0.436 \leq y \leq 0.436 \\ 0, & \text{otherwise} \end{cases}$$

This graph is different from both graphs we drew above.

Since the conditional probability $f_Y(y|x)$ depends on $x$ in this case, we can say that the random variables $X$ and $Y$ are NOT independent.

The same is true with $X$ and $Y$ in the example before with the top shaped $f(x,y)$. 
**Lemma:** Independent random variables. If $f_{XY}(x,y)$ does not depend on $y$, then $f_{X|Y}(x|y) = g(x)$ and also $f(x,y) = g(x)h(y)$.

**Proof:** By definition of $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{h(y)}$ we get

$$f(x,y) = f_{X|Y}(x|y)h(y).$$

Also by defn. substitute

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y)h(y)dy$$

$$= f_{X|Y}(x|y) \int_{-\infty}^{\infty} h(y)dy$$

since $f_{X|Y}(x|y)$ does not depend on $y$.

$$= f_{X}(x|y)$$

Same proof to show $f_{Y|X}(y|x) = h(y)$ if $f_{Y}(y|x)$ does not depend on $x$.

$X$ and $Y$ independent random variables

$\iff$

a) $f(x,y) = g(x)h(y)$

b) $f_{X|Y}(x|y) = g(x)$

c) $f_{Y|X}(y|x) = h(y)$

For all $(x,y)$

* Showing one of $a, b$ or $c$ holds for all $(x,y)$ is enough to prove $X, Y$ independent

* Showing one of $a, b$ or $c$ does not hold for some $(x,y)$ is enough to show $X, Y$ dependent

* If $X, Y$ independent then all of $a, b$ and $c$ hold.
Example: Uniform distribution

\[ f(x, y) = \begin{cases} \frac{1}{AB} & \text{if } 0 \leq x \leq A \text{ and } 0 \leq y \leq B \\ 0 & \text{elsewhere} \end{cases} \]

**NOTE:** The definition of \( f_y(y|x) \) requires \( g(x) > 0 \) so \( f_y(y|x) \) exists only for \( 0 \leq x \leq A \) in this example.

So when we say the cross-section is the same for all \( x \), we really mean for all \( x \) in the range \( 0 \leq x \leq A \).

The cross-section is the same for all \( x \) meaning \( f_y(y|x) \) does not depend on \( x \) which in turn means \( X, Y \) independent.

Let's prove it.

**Marginal densities:**

\[ g(x) = \int_{-\infty}^{\infty} p(x,y) \, dy = \int_{0}^{B} \frac{1}{AB} \, dy = \frac{1}{A} \]

\[ \text{if } 0 \leq x \leq A \]

\[ = \frac{1}{A} \]

\[ \text{otherwise} \]

\[ g(x) = \int_{-\infty}^{\infty} 0 \, dy = 0 \]

so \( g(x) = \begin{cases} \frac{1}{A} & 0 \leq x \leq A \\ 0 & \text{otherwise} \end{cases} \)

\( h(y) = \int_{-\infty}^{\infty} p(x,y) \, dx = \begin{cases} \int_{0}^{A} \frac{1}{AB} \, dx = \frac{1}{B} & 0 \leq y \leq B \\ \int_{-\infty}^{\infty} 0 \, dx = 0 & \text{elsewhere} \end{cases} \)

\[ g(x)h(y) = \begin{cases} \frac{1}{A} \cdot \frac{1}{B} & 0 \leq x \leq A \text{, and } 0 \leq y \leq B \\ 0 & \text{elsewhere} \end{cases} \]

This is exactly \( f(x,y) \).

INDEPENDENT.
Example: Let $X$ and $Y$ denote the position of an electron in the 2 dimensional Cartesian plane. Due to the uncertainty principle $X$ and $Y$ can’t be measured exactly and are random variables. You are told that the measurement along the $X$-axis is independent from the measurement along the $Y$-axis. Furthermore, let $X$ have a normal marginal density function with $\mu_X$, $\sigma_X$ and let $Y$ have a normal marginal density function with $\mu_Y$, $\sigma_Y$. What is the joint density function for $X$, $Y$?

Solution: The marginal density function for $X$ is

$$g(x) = \frac{1}{\sqrt{2\pi \sigma_X}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

The marginal density function for $Y$ is

$$h(y) = \frac{1}{\sqrt{2\pi \sigma_Y}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

Using independence, we have $f(x, y) = g(x)h(y)$, so:

$$f(x, y) = \frac{1}{\sqrt{2\pi \sigma_X}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\sqrt{2\pi \sigma_Y}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

$$= \frac{1}{2\pi \sigma_X \sigma_Y} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

If we have $\sigma_X = \sigma = \sigma$, the joint density simplifies to

$$f(x, y) = \frac{1}{2\pi \sigma^2} e^{-\frac{(x-\mu_X)^2 + (y-\mu_Y)^2}{2\sigma^2}}$$
Here is what the joint density function $f(x, y)$ looks like for $\mu_X = 1$, $\mu_Y = 2$ and $\sigma_X = \sigma_Y = 1$.

\[ f(x, y) \quad \text{Contours of constant probability.} \]

Here is what the joint density function $f(x, y)$ looks like for $\mu_X = 1$, $\mu_Y = 2$ and $\sigma_X = 0.3$, $\sigma_Y = 1$.

\[ f(x, y) \quad \text{Contours of constant probability.} \]

In this case there is more uncertainty in the $Y$ position than the $X$ position.
Example: (Example 3.19 textbook)

\[ f(x,y) = \begin{cases} 
10xy^2, & 0 < x < y < 1 \\
0, & \text{elsewhere}
\end{cases} \]

a) Find the marginal densities \( g(x) \) and the conditional density \( f_y(y|x) \).

Let's first sketch \( f(x,y) \)

\[
g(x) = \int_{-\infty}^{\infty} f(x,y) \, dy
\]

\[
= \int_{-\infty}^{x} 10xy^2 \, dy
\]

\[
= 10 \frac{xy^3}{3} \bigg|_x^1
= \frac{10x(1-x^3)}{3}
\]

Point A: \( X = Y \) from slope of line connecting \((0,0)\) to \((1,1)\)

Area of shaded cross-section is \( g(x) \) for a particular \( x \)

\[
g(x) = \begin{cases} 
\frac{10}{3} x(1-x^3), & 0 < x < 1 \\
0, & \text{elsewhere}
\end{cases}
\]
\[ h(y) = \int_{-\infty}^{\infty} f(x,y) \, dx \]
\[ = \int_{-\infty}^{y} 10xy^2 \, dx \]
\[ = 5xy^3 \bigg|_{0}^{y} \quad \text{for } 0 < y < 1 \]
\[ = 5y^4 \]
\[ \text{Area of shaded cross-section is } h(y) \text{ for that particular } y \]
\[ h(y) = \begin{cases} 
5y^4, & 0 < y < 1 \\
0, & \text{elsewhere} 
\end{cases} \]

\[ f_y(y \mid x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3} x(1-x^3)} = \frac{3y^2}{1-x^3} \text{ for } 0 < x < y < 1 \]

Notice that \( f_y(y \mid x) \) depends on \( x \) so \( X \) and \( Y \) are NOT independent. Equivalently, could show the same from \( f(x,y) \neq g(x) h(y) \).

b) Find the probability that \( Y > \frac{1}{2} \) given \( X = 0.25 \):

\[ P(Y > \frac{1}{2} \mid X = 0.25) = \int_{-\infty}^{\infty} f_y(y \mid x = 0.25) \, dy \]
\[ = \int_{-\infty}^{1/2} \frac{3y^2}{1 - 0.25^3} \, dy = \frac{5}{6} \]
Note:
\[
P(a < X < b \mid Y = y) = \int_a^b f_x(x \mid y) \, dx
\]
\[
P(a < Y < b \mid X = x) = \int_a^b f_y(y \mid x) \, dy
\]

Example 3.20 from textbook

\[
f(x, y) = \begin{cases} 
\frac{x(1+3y^2)}{4}, & 0 < x < 2, 0 < y < 1 \\
0, & \text{elsewhere}
\end{cases}
\]

(a) Find the marginal densities \(g(x), h(y)\) and the conditional density \(f_x(x \mid y)\).

Instead of sketching \(f(x, y)\) let's sketch its footprint:

\[
g(x) = \int_0^1 f(x, y) \, dy = \int_0^1 \frac{x(1+3y^2)}{4} \, dy
\]

\[
= \frac{x}{4} \left( y \bigg|_0^1 + \frac{y^3}{3} \bigg|_0^1 \right) = \frac{x}{2} \quad \text{for } 0 < x < 2
\]

Outside \(0 < x < 2\), \(g(x) = 0\)
\[ h(y) = \int_{-\infty}^{\infty} P(x,y) \, dx = \int_{0}^{2} \frac{x(1+3y^2)}{4} \, dx \]

\[ = \frac{1+3y^2}{2} \left[ \frac{x^2}{2} \right]_{0}^{2} \]

\[ = \frac{1+3y^2}{2}, \quad 0 \leq y < 1 \]

Notice \( P(x,y) = g(x) \cdot h(y) \) which means \( X \) and \( Y \) are independent.

\[ f_X(x|y) = g(x) \] since \( X \) and \( Y \) are independent.

b) Compute the probability that \( X \) is between \( \frac{1}{4} \) and \( \frac{1}{2} \) given that \( Y = \frac{1}{3} \).

\[ P\left( \frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3} \right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_X(x|y=\frac{1}{3}) \, dx \]

\[ = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{\frac{1}{2}} \, dx = \left[ \frac{x^2}{2} \right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{4} - \frac{1}{16} = \frac{3}{64} \]

\[ f_X(x|y) \] we integrate \( f_X(x|y=\frac{1}{3}) \) in this range to get

\[ P\left( \frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3} \right) \]