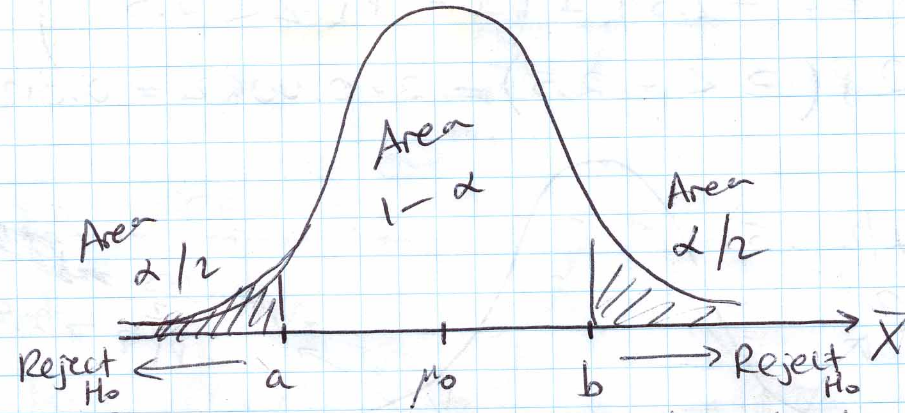
Tests concerning sample mean (Variance known)

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

Sample $X_1, \dots, X_n$
Sample mean $\bar{X}$
Known population variance σ^2

Under H_0 $\mu = \mu_0$ so P(type 1 error) is computed using the sampling distribution of $\bar{X}$ which is normal due to the central limit theorem with mean μ and standard deviation $\sigma/\sqrt{n}$. From confidence intervals we know that

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha$$



Therefore, to design a test at the level of significance α choose the critical values a and b as

$$a = \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$b = \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Then collect sample, compute the sample mean $\bar{X}$ reject H_0 if $\bar{X} < a$ OR $\bar{X} > b$

Steps in hypothesis testing

1. State the null and alternative hypothesis
2. Choose a significance level α
3. Choose test statistic and establish critical region
4. Collect sample, compute test statistic on sample
   Reject H_0 if test statistic in critical region.
   Otherwise do not reject H_0

Example: A batch of 100 resistors have an average of 102 Ohms. Assuming a population standard deviation of 8 Ohms. Test whether the population mean is 100 ohms at a significance level $\alpha = 0.05$.

Soln: (1) $H_0 : \mu = \underbrace{100}_{\text{This is } \mu_0}$ $\quad H_1 : \mu \neq 100$

(Note unless otherwise stated we use a two-tailed test)

(2) $\alpha = 0.05$

(3) Test statistic $\bar{X}$, Reject H_0 if

$$\bar{X} < a \quad \text{or} \quad \bar{X} > b$$

$$\boxed{z_{\alpha/2} = z_{0.025} = 1.96}$$

$$a = \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 100 - 1.96 \frac{8}{10}$$
$$= 98.432$$

$$b = \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 100 + 1.96 \frac{8}{10}$$
$$= 101.568$$

(4) $\bar{X} = 102 > b$ therefore

Reject H_0.

One-sided tests of the sample mean

(A) $H_0 : \mu = \mu_0$ $\quad$ Reject H_0 at significance level α

$H_1 : \mu > \mu_0$ $\quad$ if $\bar{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$

Note that z_α appears instead of $z_{\alpha/2}$ just like in one-tailed confidence intervals

(B) $H_0 : \mu = \mu_0$ $\quad$ Reject H_0 at a significance level

$H_1 : \mu < \mu_0$ $\quad$ α if $\bar{X} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$

Example: A quality control engineer finds that a sample of 100 light bulbs had an average life-time of 470 hours. Assuming a population standard deviation $\sigma = 25$ hours, test whether the population mean is 480 hours vs. the alternative hypothesis $\mu < 480$ at a significance level $\alpha = 0.05$

Soln: (1) $H_0 : \mu = 480$ $\quad H_1 : \mu < 480$ (one-tailed test)

(2) $\alpha = 0.05$

(3) Test statistic $\bar{X}$. Reject H_0 if

$$\bar{X} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 480 - 1.645 \frac{25}{10}$$

$$\bar{X} < 475.9$$

(4) Since $\bar{X} = 470$ Reject H_0

Note: Table A-3 has no entry with $P(Z < z) = 0.05$ exactly. The closest ones are $P(Z < -1.64) = 0.505$ and $P(z < -1.65) = 0.495$

The book (and in this example we) use the average of 1.64 and 1.65 as the z-value for which $P(Z > z) = 0.05$. Alternatively, we could just use whichever is closest to the α value we are looking for.

Tests concerning sample mean (Variance unknown)

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

Sample $X_1, \ldots, X_n$ from a normal population. Unknown σ^2.
Sample mean $\bar{X}$, sample variance S^2

We know that in this case the sampling distribution for $\bar{X}$ is the t-distribution.

Critical region at significance level α is

$$\bar{X} < a \quad \text{OR} \quad \bar{X} > b \quad \text{reject } H_0$$

where
$$a = \mu_0 - t_{\alpha/2} \frac{S}{\sqrt{n}}$$
$$b = \mu_0 + t_{\alpha/2} \frac{S}{\sqrt{n}}$$
where $t_{\alpha/2}$ is from row with $v = n-1$

OR equivalently let $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$

reject H_0 if $T < -t_{\alpha/2}$ or $T > t_{\alpha/2}$ for $v = n-1$ degrees of freedom.

In one-sided test $t_{\alpha/2}$ is replaced by t_α as usual.

<u>Example</u> (Example 10.5 textbook)

Claim: A certain electrical appliance (vacum cleaner) expends 46 kWh per year. A random sample of 12 homes indicate that vacum cleaners expend an average of 42 kWh per year with standard deviation 11.9 kWh. Does this suggest at the 0.05 level of significance that, on the average, vacum cleaners expend less than 46 kWh per year? Assume population of kWh to be normally distributed.

<u>Soln:</u> (1) $H_0: \mu = 46$ kWh

$H_1: \mu < 46$ kWh

(2) $\alpha = 0.05$

(3) Test statistic

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

reject H_0 if

$T < -t_{0.05}$

for $v = n-1 = 11$ degrees of freedom.

$t_{0.05} = 1.796$

so reject if

$T < -1.796$

(4) $\bar{X} = 42$ $s = 11.9$ $n = 12$

$$T = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16$$

Do not reject H_0

P-values

Preselection of significance level α has its roots in the philosophy that making a type-I error should be controlled.

The P-value approach aims to give more information to the user, especially when the test-statistic is close to the critical region.

<u>Defn</u>: A P-value is the lowest level of significance at which the observed value of a test statistic is significant (rejects H_0).

<u>Example</u>: A batch of 100 resistors have an average of 101.5 Ohms. Assuming a population standard deviation of 5 Ohms

a) Test whether the population mean is 100 Ohms at level of significance $\alpha = 0.05$

b) Compute p-value.

<u>Soln</u> a) $H_0 : \mu = 100 \quad H_1 : \mu \neq 100$

Test statistic $\bar{X}$. Reject H_0 if

$$\bar{X} < 100 - z_{0.025} \frac{\sigma}{\sqrt{n}} = 100 - 1.96 \times \frac{5}{10}$$
$$= 99.02$$

OR

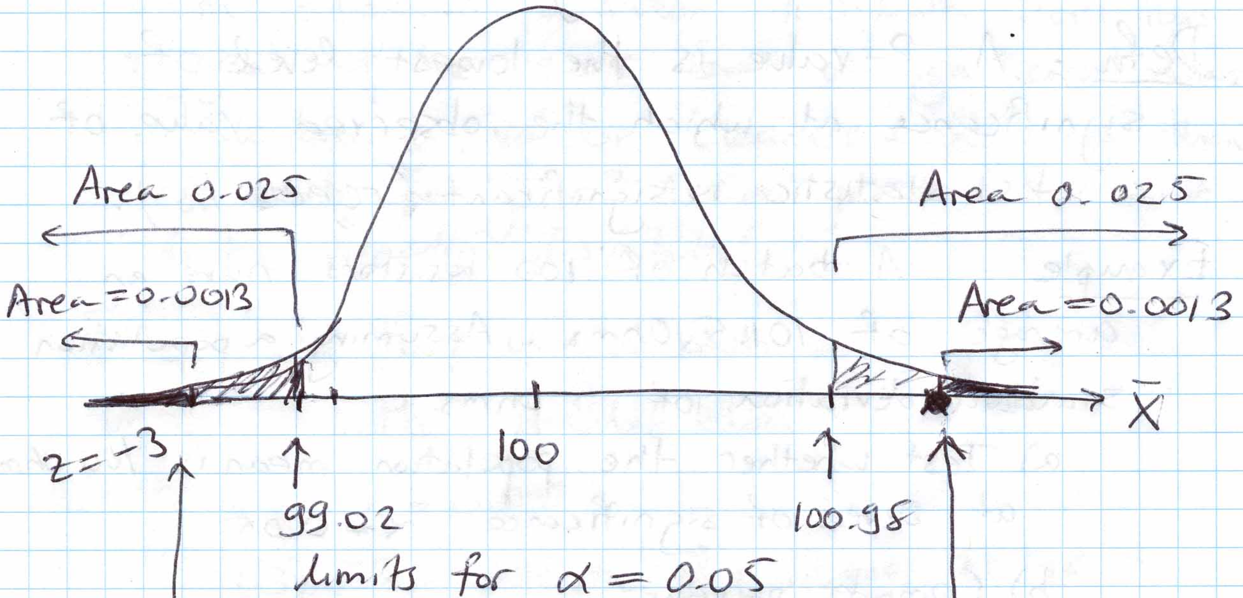
$$\bar{X} > 100 + z_{0.025} \frac{\sigma}{\sqrt{n}} = 100 + 1.96 \times \frac{5}{10}$$
$$= 100.98$$

$\bar{X} = 101.5$ therefore reject H_0

b) $$z = \frac{\bar{X} - 100}{\sigma/\sqrt{n}} = \frac{101.5 - 100}{5/10} = 3$$ observed z-value

$$P = 2P(Z > 3 \text{ when } \mu = 100) = 2 \times 0.0013 = 0.0026$$

This means that H_0 could have been rejected at significance level $\alpha = 0.0026$ which is much stronger than rejecting it at $\alpha = 0.05$. Hence, the P-value gives more information than rejecting or not rejecting H_0 at a fixed α.



(A) Observed $\bar{X} = 101.5$

$z = 3$

very rare event

Could have moved critical value all the way here and still reject H_0

(B) Since this is a two-tailed test, the critical value on this side mirrors the right tail

⇓

(C) Therefore

$P = 2 \times 0.0013 =$

0.0026